



**Pedro Paiva Zühlke d'Oliveira**

**Homotopies of Curves on the 2-Sphere with  
Geodesic Curvature in a Prescribed Interval**

**Tese de Doutorado**

Thesis presented to the Programa de Pós-Graduação em Matemática of the Departamento de Matemática, PUC-Rio as partial fulfillment of the requirements for the degree of Doutor em Matemática.

Advisor: Prof. Nicolau Corção Saldanha

Rio de Janeiro  
September 2012



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Rio de Janeiro, 10/09/2012

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Bibliographic data

Zühlke, Pedro

Homotopies of Curves on the 2-Sphere with Geodesic Curvature in a Prescribed Interval / Pedro Paiva Zühlke d'Oliveira ; advisor: Nicolau Corção Saldanha. — 2012.

117 f. : il. ; 30 cm

Tese (Doutorado em Matemática)-Pontifícia Universidade Católica do Rio de Janeiro, Rio de Janeiro, 2012.

Inclui bibliografia

1. Matemática – Teses. 2. Curve. 3. Curvature. 4. Geometry. 5. Homotopy. 6. Topology. I. Saldanha, Nicolau C.. II. Pontifícia Universidade Católica do Rio de Janeiro. Departamento de Matemática. III. Título.

CDD: 510

## Acknowledgments

I thank prof. Nicolau C. Saldanha for his support, patience and immense generosity. All of the results in this work bear his influence in some form. I feel privileged to be his student and friend. I also thank profs. Alexei N. Krasilnikov and Paul A. Schweitzer, S.J., for the kindness and generosity with which they have always treated me. Without the help of all three, I would hardly have obtained a PhD degree.

During the last few years I was partially supported by scholarships (from CNPq and CAPES); I would like to thank everyone who worked to make them available to me.

## Abstract

Zühlke, Pedro; Saldanha, Nicolau C.. **Homotopies of Curves on the 2-Sphere with Geodesic Curvature in a Prescribed Interval**. Rio de Janeiro, 2012. 117p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

For  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ , let  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  denote the set of all closed curves of class  $C^r$  on the sphere  $\mathbf{S}^2$  whose geodesic curvatures lie in the interval  $(\kappa_1, \kappa_2)$ , furnished with the  $C^r$  topology (for some  $r \geq 2$ ). In 1970, J. Little proved that the space  $\mathcal{L}_0^{+\infty}$  of closed curves having positive geodesic curvature has three connected components. Let  $\rho_i = \operatorname{arccot} \kappa_i$  ( $i = 1, 2$ ). In this thesis, we show that  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  has  $n$  connected components  $\mathcal{L}_1, \dots, \mathcal{L}_n$ , where

$$n = \left\lfloor \frac{\pi}{\rho_1 - \rho_2} \right\rfloor + 1$$

and  $\mathcal{L}_j$  contains circles traversed  $j$  times ( $1 \leq j \leq n$ ). The component  $\mathcal{L}_{n-1}$  also contains circles traversed  $(n-1) + 2k$  times, and  $\mathcal{L}_n$  contains circles traversed  $n + 2k$  times, for any  $k \in \mathbf{N}$ . In addition, each of  $\mathcal{L}_1, \dots, \mathcal{L}_{n-2}$  is homotopy equivalent to  $\mathbf{SO}_3$  ( $n \geq 3$ ). A direct characterization of the components in terms of the properties of a curve and a proof that  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  is homeomorphic to  $\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}$  whenever  $\rho_1 - \rho_2 = \bar{\rho}_1 - \bar{\rho}_2$  ( $\bar{\rho}_i = \operatorname{arccot} \bar{\kappa}_i$ ) are also presented.

## Keywords

Curve. Curvature. Geometry. Homotopy. Topology.

## Resumo

Zühlke, Pedro; Saldanha, Nicolau C.. **Homotopias de Curvas na Esfera com Curvatura Geodésica num Intervalo Dado**. Rio de Janeiro, 2012. 117p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Para  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ , seja  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  o conjunto de todas as curvas fechadas de classe  $C^r$  na esfera  $\mathbf{S}^2$  cujas curvaturas geodésicas estão restritas ao intervalo  $(\kappa_1, \kappa_2)$ , munido da topologia  $C^r$  (para algum  $r \geq 2$ ). Em 1970, J. Little provou que o espaço  $\mathcal{L}_0^{+\infty}$  de curvas fechadas com curvatura geodésica positiva possui três componentes conexas. Sejam  $\rho_i = \operatorname{arccot} \kappa_i$  ( $i = 1, 2$ ). Nesta tese, mostramos que  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  possui  $n$  componentes conexas  $\mathcal{L}_1, \dots, \mathcal{L}_n$ , onde

$$n = \left\lfloor \frac{\pi}{\rho_1 - \rho_2} \right\rfloor + 1$$

e  $\mathcal{L}_j$  contém círculos percorridos  $j$  vezes ( $1 \leq j \leq n$ ). A componente  $\mathcal{L}_{n-1}$  também contém círculos percorridos  $(n-1) + 2k$  vezes, e  $\mathcal{L}_n$  contém círculos percorridos  $n + 2k$  vezes, para qualquer  $k \in \mathbf{N}$ . Além disto,  $\mathcal{L}_1, \dots, \mathcal{L}_{n-2}$  são todos homotopicamente equivalentes a  $\mathbf{SO}_3$  ( $n \geq 3$ ). Também são exibidas uma caracterização das componentes em termos das propriedades de uma curva e uma prova de que  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  é homeomorfo a  $\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}$  se  $\rho_1 - \rho_2 = \bar{\rho}_1 - \bar{\rho}_2$  ( $\bar{\rho}_i = \operatorname{arccot} \bar{\kappa}_i$ ).

## Palavras-chave

Curva. Curvatura. Geometria. Homotopia. Topologia.

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# 1

## Introduction

### History of the problem

Consider the set  $\mathcal{W}$  of all  $C^r$  regular closed curves in the plane  $\mathbf{R}^2$  (i.e.,  $C^r$  immersions  $\mathbf{S}^1 \rightarrow \mathbf{R}^2$ ), furnished with the  $C^r$  topology ( $r \geq 1$ ). The Whitney-Graustein theorem ([17], thm. 1) states that two such curves are homotopic through regular closed curves if and only if they have the same rotation number (where the latter is the number of full turns of the tangent vector to the curve).<sup>1</sup> Thus, the space  $\mathcal{W}$  has an infinite number of connected components  $\mathcal{W}_n$ , one for each rotation number  $n \in \mathbf{Z}$ . A typical element of  $\mathcal{W}_n$  ( $n \neq 0$ ) is a circle traversed  $|n|$  times, with the direction depending on the sign of  $n$ ;  $\mathcal{W}_0$  contains a figure eight curve.

For curves on the unit sphere  $\mathbf{S}^2 \subset \mathbf{R}^3$ , there is no natural notion of rotation number. Indeed, the corresponding space  $\mathcal{I}$  of  $C^r$  immersions  $\mathbf{S}^1 \rightarrow \mathbf{S}^2$  (i.e., regular closed curves on  $\mathbf{S}^2$ ) has only two connected components  $\mathcal{I}_+$  and  $\mathcal{I}_-$ ; this is an immediate consequence of a much more general result of S. Smale ([16], thm. A). The component  $\mathcal{I}_-$  contains all circles traversed an odd number of times, and the component  $\mathcal{I}_+$  contains all circles traversed an even number of times. Actually, the Hirsch-Smale theorem implies that  $\mathcal{I}_\pm \simeq \mathbf{SO}_3 \times \Omega\mathbf{S}^3$ , where  $\Omega\mathbf{S}^3$  denotes the set of all continuous closed curves on  $\mathbf{S}^3$ , with the compact-open topology; the properties of the latter space are well understood (see [1], §16).<sup>2</sup>

In 1970, J. A. Little formulated and solved the following problem: Let  $\mathcal{L}$  denote the set of all  $C^2$  closed curves on  $\mathbf{S}^2$  which have nonvanishing geodesic curvature, with the  $C^2$  topology; what are the connected components of  $\mathcal{L}$ ? Although his motivation to investigate  $\mathcal{L}$  appears to have been purely geometric, this space arises naturally in the study of a certain class of linear ordinary differential equations (see [12] for a discussion of this class and further references).

<sup>1</sup>Numbers enclosed in brackets refer to works listed in the bibliography at the end.

<sup>2</sup>The notation  $X \simeq Y$  (resp.  $X \approx Y$ ) means that  $X$  is homotopy equivalent (resp. homeomorphic) to  $Y$ .



Little was able to show (see [8], thm. 1) that  $\mathcal{L}$  has six connected components,  $\mathcal{L}_{\pm 1}$ ,  $\mathcal{L}_{\pm 2}$  and  $\mathcal{L}_{\pm 3}$ , where the sign indicates the sign of the geodesic curvature of a curve in the corresponding component. A homeomorphism between  $\mathcal{L}_i$  and  $\mathcal{L}_{-i}$  is obtained by reversing the orientation of the curves in  $\mathcal{L}_i$ .

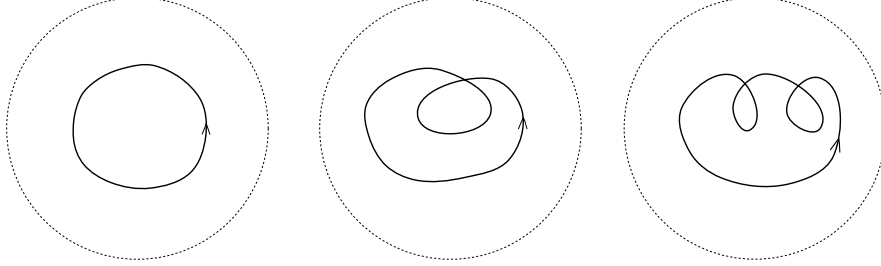


Figure 1: The curves depicted above provide representatives of the components  $\mathcal{L}_1$ ,  $\mathcal{L}_2$  and  $\mathcal{L}_3$ , respectively. All three are contained in the upper hemisphere of  $\mathbf{S}^2$ ; the dashed line represents the equator seen from above.

The topology of the space  $\mathcal{L}$  has been investigated by quite a few other people since Little. We mention here only B. Khesin, B. Shapiro and M. Shapiro, who studied  $\mathcal{L}$  and similar spaces in the 1990's (cf. [6], [7], [14] and [15]). They showed that  $\mathcal{L}_{\pm 1}$  are homotopy equivalent to  $\mathbf{SO}_3$ , and also determined the number of connected components of the spaces analogous to  $\mathcal{L}$  in  $\mathbf{R}^n$ ,  $\mathbf{S}^n$  and  $\mathbf{RP}^n$ , for arbitrary  $n$ .

The first pieces of information about the homotopy and cohomology groups  $\pi_k(\mathcal{L})$  and  $H^k(\mathcal{L})$  for  $k \geq 1$  were, however, only obtained a decade later by N. C. Saldanha in [10] and [11]. Finally, in the recent work [12], Saldanha gave a complete description of the homotopy type of  $\mathcal{L}$  and other closely related spaces of curves on  $\mathbf{S}^2$ . He proved in particular that

$$\begin{aligned}\mathcal{L}_{\pm 2} &\simeq \mathbf{SO}_3 \times (\Omega \mathbf{S}^3 \vee \mathbf{S}^2 \vee \mathbf{S}^6 \vee \mathbf{S}^{10} \vee \dots) \quad \text{and} \\ \mathcal{L}_{\pm 3} &\simeq \mathbf{SO}_3 \times (\Omega \mathbf{S}^3 \vee \mathbf{S}^4 \vee \mathbf{S}^8 \vee \mathbf{S}^{12} \vee \dots).\end{aligned}$$

The reason for the appearance of an  $\mathbf{SO}_3$  factor in all of these results is that (unlike Saldanha, cf. [12]) we have not chosen a basepoint for the unit tangent bundle  $UT\mathbf{S}^2 \approx \mathbf{SO}_3$ ; a careful discussion of this is given in §1.

## Overview of this work

The main purpose of this thesis is to generalize Little's theorem to other spaces of closed curves on  $\mathbf{S}^2$ . Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$  be given and let  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  be the set of all  $C^r$  closed curves on  $\mathbf{S}^2$  whose geodesic curvatures are

restricted to lie in the interval  $(\kappa_1, \kappa_2)$ , furnished with the  $C^r$  topology (for some  $r \geq 2$ ); in this notation, the spaces  $\mathcal{L}$  and  $\mathcal{J}$  discussed above become  $\mathcal{L}_{-\infty}^0 \sqcup \mathcal{L}_0^{+\infty}$  and  $\mathcal{L}_{-\infty}^{+\infty}$ , respectively. We present a direct characterization of the connected components of  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  in terms of the pair  $\kappa_1 < \kappa_2$  and of the properties of curves in  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ . It is shown in particular that the number of components is always finite, and a simple formula for it in terms of  $\kappa_1$  and  $\kappa_2$  is deduced.

More precisely, let  $\rho_i = \operatorname{arccot}(\kappa_i)$ ,  $i = 1, 2$ , where we adopt the convention that  $\operatorname{arccot}$  takes values in  $[0, \pi]$ , with  $\operatorname{arccot}(+\infty) = 0$  and  $\operatorname{arccot}(-\infty) = \pi$ . Also, let  $\lfloor x \rfloor$  denote the greatest integer smaller than or equal to  $x$ . Then  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  has  $n$  connected components  $\mathcal{L}_1, \dots, \mathcal{L}_n$ , where

$$n = \left\lfloor \frac{\pi}{\rho_1 - \rho_2} \right\rfloor + 1$$

and  $\mathcal{L}_j$  contains circles traversed  $j$  times ( $1 \leq j \leq n$ ). The component  $\mathcal{L}_{n-1}$  also contains circles traversed  $(n-1) + 2k$  times, and  $\mathcal{L}_n$  contains circles traversed  $n + 2k$  times, for  $k \in \mathbf{N}$ . In addition, it will be seen that each of  $\mathcal{L}_1, \dots, \mathcal{L}_{n-2}$  is homotopy equivalent to  $\mathbf{SO}_3$  ( $n \geq 3$ ).

This result could be considered a first step towards the determination of the homotopy type of  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  in terms of  $\kappa_1$  and  $\kappa_2$ . In this context, it is natural to ask whether the inclusion  $\mathcal{L}_{\kappa_1}^{\kappa_2} \hookrightarrow \mathcal{L}_{-\infty}^{+\infty} = \mathcal{J}$  is a homotopy equivalence; as we have already mentioned, the topology of the latter space is well understood. It will be shown that the answer is negative when  $\rho_1 - \rho_2 \leq \frac{2\pi}{3}$ . We expect this to be false except when  $\kappa_1 = -\infty$  and  $\kappa_2 = +\infty$ . Actually, we conjecture that  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  and  $\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}$  have different homotopy types if and only if  $\rho_1 - \rho_2 \neq \bar{\rho}_1 - \bar{\rho}_2$ , but here it will only be proved that  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  is homeomorphic to  $\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}$  if  $\rho_1 - \rho_2 = \bar{\rho}_1 - \bar{\rho}_2$  ( $\rho_i = \operatorname{arccot} \kappa_i$  and  $\bar{\rho}_i = \operatorname{arccot} \bar{\kappa}_i$ ).

## Brief outline of the sections

It turns out that it is more convenient, but not essential, to work with curves which need not be  $C^2$ . The curves that we consider possess continuously varying unit tangent vectors at all points, but their geodesic curvatures are defined only almost everywhere. This class of curves is described in §1, where we also relate the resulting spaces of curves to the more familiar spaces of  $C^r$  curves. In this section we take the first steps toward the main theorem by proving that the topology of  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  depends only on  $\rho_1 - \rho_2$ . A corollary of this result is that any space  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  is homeomorphic to a space of type  $\mathcal{L}_{\kappa_0}^{+\infty}$ ; the latter class is usually more convenient to work with. Some variations of our definition are also investigated. In particular, in this section we consider spaces

of non-closed curves.

In §2, we study curves which have image contained in a hemisphere. Almost all of this section is dedicated to proof that it is possible to assign to each such curve a distinguished hemisphere  $h_\gamma$  containing its image, in such a way that  $h_\gamma$  depends continuously on  $\gamma$ .

The main tools in the thesis are introduced in §3. Given a curve  $\gamma$ , we assign to  $\gamma$  certain maps  $B_\gamma$  and  $C_\gamma$ , called the regular and caustic bands spanned by  $\gamma$ , respectively. These are “fat” versions of the curve, and each of them carries in geometric form important information on the curve. We separate our curves into two main classes, called condensed and diffuse, depending on the properties of its caustic band. This distinction is essential throughout the work.

In §4, the grafting construction is explained. If the curve is diffuse, then we can use grafting to deform it into a circle traversed a certain number of times, which is the canonical curve in our spaces. We reach the same conclusion for condensed curves, using very different methods, in §5, where a notion of rotation numbers for curves of this type is also introduced. Although there exist curves which are neither condensed nor diffuse, any such curve is homotopic to a curve of one of these two types. The main results used to establish this are presented in §6.

In §7, we decide when it is possible to deform a circle traversed  $k$  times into a circle traversed  $k + 2$  times in  $\mathcal{L}_{\kappa_0}^{+\infty}$ . It is seen that this is possible if and only if  $k \geq n - 1 = \left\lfloor \frac{\pi}{\rho_0} \right\rfloor$  (where  $\rho_0 = \operatorname{arccot} \kappa_0$ ), and an explicit homotopy when this is the case is presented. It is also shown that the set of condensed curves in  $\mathcal{L}_{\kappa_0}^{+\infty}$  with fixed rotation number  $k < n - 1$  is a connected component of this space.

The proofs of the main theorems are given in §8, after most of the work has been done. A direct characterization of the components of  $\mathcal{L}_{\kappa_0}^{+\infty}$  ( $\kappa_0 \in \mathbf{R}$ ) in terms of the properties of a curve is presented at the end of this section.

The last section is dedicated to the proof that the inclusion  $\mathcal{L}_{\kappa_1}^{\kappa_2} \hookrightarrow \mathcal{L}_{-\infty}^{+\infty} = \mathcal{J}$  is not a (weak) homotopy equivalence if  $\rho_1 - \rho_2 \leq \frac{2\pi}{3}$ .

Finally, we present in an appendix some basic results on convexity in  $\mathbf{S}^n$  that are used throughout the thesis. Although none of these results is new, complete proofs are given.

## 2

## Spaces of Curves of Bounded Geodesic Curvature

### Basic definitions and notation

Let  $M$  denote either the euclidean space  $\mathbf{R}^{n+1}$  or the unit sphere  $\mathbf{S}^n \subset \mathbf{R}^{n+1}$ , for some  $n \geq 1$ . By a *curve*  $\gamma$  in  $M$  we mean a continuous map  $\gamma: [a, b] \rightarrow M$ . A curve will be called *regular* when it has a continuous and nonvanishing derivative; in other words, a regular curve is a  $C^1$  immersion of  $[a, b]$  into  $M$ . For simplicity, the interval where  $\gamma$  is defined will usually be  $[0, 1]$ .

Let  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  be a regular curve and let  $|\cdot|$  denote the usual Euclidean norm. The *arc-length parameter*  $s$  of  $\gamma$  is defined by

$$s(t) = \int_0^t |\dot{\gamma}(t)| dt,$$

and  $L = \int_0^1 |\dot{\gamma}(t)| dt$  is called the *length* of  $\gamma$ . Since  $\dot{s}(t) > 0$  for all  $t$ ,  $s$  is an invertible function, and we may parametrize  $\gamma$  by  $s \in [0, L]$ . Derivatives with respect to  $t$  and  $s$  will be systematically denoted by a  $\dot{\phantom{x}}$  and a  $'$ , respectively; this convention extends, of course, to higher-order derivatives as well.

Up to homotopy, we can always assume that a family of curves is parametrized proportionally to arc-length.

**(2.1) Lemma.** *Let  $A$  be a topological space and let  $a \mapsto \gamma_a$  be a continuous map from  $A$  to the set of all  $C^r$  regular curves  $\gamma: [0, 1] \rightarrow M$  ( $r \geq 1$ ) with the  $C^r$  topology. Then there exists a homotopy  $\gamma_a^u: [0, 1] \rightarrow M$ ,  $u \in [0, 1]$ , such that for any  $a \in A$ :*

(i)  $\gamma_a^0 = \gamma_a$  and  $\gamma_a^1$  is parametrized so that  $|\dot{\gamma}_a^1(t)|$  is independent of  $t$ .

(ii)  $\gamma_a^u$  is an orientation-preserving reparametrization of  $\gamma_a$ , for all  $u \in [0, 1]$ .

*Proof.* Let  $s_a(t) = \int_0^t |\dot{\gamma}_a(\tau)| d\tau$  be the arc-length parameter of  $\gamma_a$ ,  $L_a$  its length and  $\tau_a: [0, L_a] \rightarrow [0, 1]$  the inverse function of  $s_a$ . Define  $\gamma_a^u: [0, 1] \rightarrow M$  by:

$$\gamma_a^u(t) = \gamma_a((1-u)t + u\tau_a(L_a t)) \quad (u, t \in [0, 1], a \in A).$$

Then  $\gamma_a^u$  is the desired homotopy.  $\square$

The unit tangent vector to  $\gamma$  at  $\gamma(t)$  will always be denoted by  $\mathbf{t}(t)$ . Set  $M = \mathbf{S}^2$  for the rest of this section, and define the *unit normal vector*  $\mathbf{n}$  to  $\gamma$  by

$$\mathbf{n}(t) = \gamma(t) \times \mathbf{t}(t),$$

where  $\times$  denotes the vector product in  $\mathbf{R}^3$ . Equivalently,  $\mathbf{n}(t)$  is the unique vector which makes  $(\gamma(t), \mathbf{t}(t), \mathbf{n}(t))$  a positively oriented orthonormal basis of  $\mathbf{R}^3$ .

Assume now that  $\gamma$  has a second derivative. By definition, the *geodesic curvature*  $\kappa(s)$  at  $\gamma(s)$  is given by

$$\kappa(s) = \langle \mathbf{t}'(s), \mathbf{n}(s) \rangle. \quad (1)$$

Note that the geodesic curvature is not altered by an orientation-preserving reparametrization of the curve, but its sign is changed if we use an orientation-reversing reparametrization. Since the sectional curvatures of the sphere are all equal to 1, the normal curvature of  $\gamma$  is 1 at each point. In particular, its *Euclidean curvature*  $K$ ,

$$K(s) = \sqrt{1 + \kappa(s)^2},$$

never vanishes.

Closely related to the geodesic curvature of a curve  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  is the *radius of curvature*  $\rho(t)$  of  $\gamma$  at  $\gamma(t)$ , which we define as the unique number in  $(0, \pi)$  satisfying

$$\cot \rho(t) = \kappa(t).$$

Note that the sign of  $\kappa(t)$  is equal to the sign of  $\frac{\pi}{2} - \rho(t)$ .

*Example.* A parallel circle of colatitude  $\alpha$ , for  $0 < \alpha < \pi$ , has geodesic curvature  $\pm \cot \alpha$  (the sign depends on the orientation), and radius of curvature  $\alpha$  or  $\pi - \alpha$  at each point. (Recall that the colatitude of a point measures its distance from the north pole along  $\mathbf{S}^2$ .) The radius of curvature  $\rho(t)$  of an arbitrary curve  $\gamma$  gives the size of the radius of the osculating circle to  $\gamma$  at  $\gamma(t)$ , measured along  $\mathbf{S}^2$  and taking the orientation of  $\gamma$  into account.

If we consider  $\gamma$  as a curve in  $\mathbf{R}^3$ , then its “usual” radius of curvature  $R$  is defined by  $R(t) = \frac{1}{K(t)} = \sin \rho(t)$ . We will rarely mention  $R$  or  $K$  again, preferring instead to work with  $\rho$  and  $\kappa$ , which are their natural intrinsic analogues in the sphere.

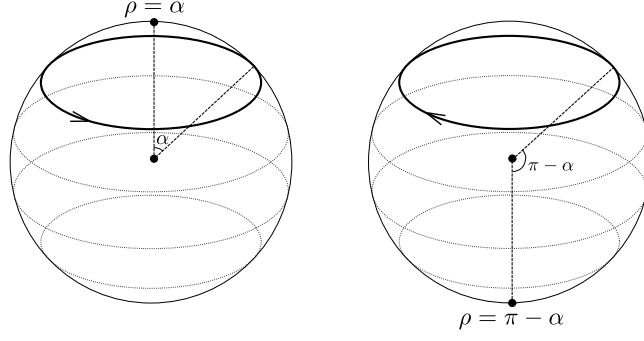


Figure 2: A parallel circle of colatitude  $\alpha$  has radius of curvature  $\alpha$  or  $\pi - \alpha$ , depending on its orientation. In the first figure the center of the circle on  $\mathbf{S}^2$  is taken to be the north pole, and in the second, the south pole.

### Spaces of curves

Given  $p \in \mathbf{S}^2$  and  $v \in T_p\mathbf{S}^2$  of norm 1, there exists a unique  $Q \in \mathbf{SO}_3$  having  $p \in \mathbf{R}^3$  as first column and  $v \in \mathbf{R}^3$  as second column. We obtain thus a diffeomorphism between  $\mathbf{SO}_3$  and the unit tangent bundle  $UT\mathbf{S}^2$  of  $\mathbf{S}^2$ .

**(2.2) Definition.** For a regular curve  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$ , its *frame*  $\Phi_\gamma: [0, 1] \rightarrow \mathbf{SO}_3$  is the map given by

$$\Phi_\gamma(t) = \begin{pmatrix} | & | & | \\ \gamma(t) & \mathbf{t}(t) & \mathbf{n}(t) \\ | & | & | \end{pmatrix}^1.$$

In other words,  $\Phi_\gamma$  is the curve in  $UT\mathbf{S}^2$  associated with  $\gamma$ , under the identification of  $UT\mathbf{S}^2$  with  $\mathbf{SO}_3$ . We emphasize that it is not necessary that  $\gamma$  have a second derivative for  $\Phi_\gamma$  to be defined.

Now let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$  and  $Q \in \mathbf{SO}_3$ . We would like to study the space  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  of all regular curves  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  satisfying:

- (i)  $\Phi_\gamma(0) = I$  and  $\Phi_\gamma(1) = Q$ ;
- (ii)  $\kappa_1 < \kappa(t) < \kappa_2$  for each  $t \in [0, 1]$ .

Here  $I$  is the 3×3 identity matrix and  $\kappa$  is the geodesic curvature of  $\gamma$ . Condition (i) says that  $\gamma$  starts at  $e_1$  in the direction  $e_2$  and ends at  $Qe_1$  in the direction  $Qe_2$ .

<sup>1</sup>In the works of Saldanha this is denoted by  $\mathfrak{F}_\gamma$  and called the *Frenet frame* of  $\gamma$ . We will not use this terminology to avoid any confusion with the usual Frenet frame of  $\gamma$  when it is considered as a curve in  $\mathbf{R}^3$ .

This definition is incomplete because we have not described the topology of  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ , nor explained what is meant by the geodesic curvature of a regular curve (which need not have a second derivative, according to our definition). The most natural choice would be to require that the curves in this space be of class  $C^2$ , and to give it the  $C^2$  topology. The foremost reason why we will not follow this course is that we would like to be able to perform some constructions which yield curves that are not  $C^2$ . For instance, we may wish to construct a curve  $\gamma$  of positive geodesic curvature by concatenating two arcs of circles  $\sigma_1$  and  $\sigma_2$  of different radii, as in fig. 3 below. Even though the resulting curve is regular, it is not possible to assign any meaningful value to the curvature of  $\gamma$  at  $p$ . However, we may approximate  $\gamma$  as well as we like by a smooth curve which does have everywhere positive geodesic curvature. We shall adopt a more complicated definition precisely in order to avoid using convolutions or other tools all the time to smoothen such a curve.

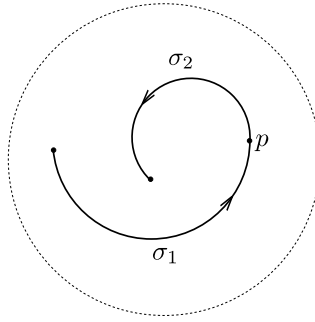


Figure 3: A curve on  $\mathbf{S}^2$  obtained by concatenation of arcs of circles of different radii. The dashed line represents the equator.

**(2.3) Definition.** A function  $f: [a, b] \rightarrow \mathbf{R}$  is said to be of class  $H^1$  if it is an indefinite integral of some  $g \in L^2[a, b]$ . We extend this definition to maps  $F: [a, b] \rightarrow \mathbf{R}^n$  by saying that  $F$  is of class  $H^1$  if and only if each of its component functions is of class  $H^1$ .

Since  $L^2[a, b] \subset L^1[a, b]$ , an  $H^1$  function is absolutely continuous (and differentiable almost everywhere).

We shall now present an explicit description of a topology on  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  which turns it into a Hilbert manifold. The definition is unfortunately not very natural. However, we shall prove the following two results relating this space to more familiar concepts: First, for any  $r \in \mathbf{N}$ ,  $r \geq 2$ , the subset of  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  consisting of  $C^r$  curves will be shown to be dense in  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ . Second,

we will see that the space of  $C^r$  regular curves satisfying conditions (i) and (ii) above, with the  $C^r$  topology, is (weakly) homotopy equivalent to  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ .<sup>2</sup>

Consider first a smooth regular curve  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$ . From the definition of  $\Phi_\gamma$  we deduce that

$$\dot{\Phi}_\gamma(t) = \Phi_\gamma(t)\Lambda(t), \quad \text{where } \Lambda(t) = \begin{pmatrix} 0 & -|\dot{\gamma}(t)| & 0 \\ |\dot{\gamma}(t)| & 0 & -|\dot{\gamma}(t)|\kappa(t) \\ 0 & |\dot{\gamma}(t)|\kappa(t) & 0 \end{pmatrix} \in \mathfrak{so}_3 \quad (2)$$

is called the *logarithmic derivative* of  $\Phi_\gamma$  and  $\kappa$  is the geodesic curvature of  $\gamma$ .

Conversely, given  $Q_0 \in \mathbf{SO}_3$  and a smooth map  $\Lambda: [0, 1] \rightarrow \mathfrak{so}_3$  of the form

$$\Lambda(t) = \begin{pmatrix} 0 & -v(t) & 0 \\ v(t) & 0 & -w(t) \\ 0 & w(t) & 0 \end{pmatrix}, \quad (3)$$

let  $\Phi: [0, 1] \rightarrow \mathbf{SO}_3$  be the unique solution to the initial value problem

$$\dot{\Phi}(t) = \Phi(t)\Lambda(t), \quad \Phi(0) = Q_0. \quad (4)$$

Define  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  to be the smooth curve given by  $\gamma(t) = \Phi(t)(e_1)$ . Then  $\gamma$  is regular if and only if  $v(t) \neq 0$  for all  $t \in [0, 1]$ , and it satisfies  $\Phi_\gamma = \Phi$  if and only if  $v(t) > 0$  for all  $t$ . (If  $v(t) < 0$  for all  $t$  then  $\gamma$  is regular, but  $\Phi_\gamma$  is obtained from  $\Phi$  by changing the sign of the entries in the second and third columns.)

Equation (4) still has a unique solution if we only require that  $v, w \in L^2[0, 1]$  (cf. [3], p. 67). With this in mind, let  $\mathbf{E} = L^2[0, 1] \times L^2[0, 1]$  and let  $h: (0, +\infty) \rightarrow \mathbf{R}$  be the smooth diffeomorphism

$$h(t) = t - t^{-1}. \quad (5)$$

For each pair  $\kappa_1 < \kappa_2 \in \mathbf{R}$ , let  $h_{\kappa_1, \kappa_2}: (\kappa_1, \kappa_2) \rightarrow \mathbf{R}$  be the smooth diffeomorphism

$$h_{\kappa_1, \kappa_2}(t) = (\kappa_1 - t)^{-1} + (\kappa_2 - t)^{-1}$$

and, similarly, set

$$\begin{aligned} h_{-\infty, +\infty}: \mathbf{R} &\rightarrow \mathbf{R} & h_{-\infty, +\infty}(t) &= t \\ h_{-\infty, \kappa_2}: (-\infty, \kappa_2) &\rightarrow \mathbf{R} & h_{-\infty, \kappa_2}(t) &= t + (\kappa_2 - t)^{-1} \\ h_{\kappa_1, +\infty}: (\kappa_1, +\infty) &\rightarrow \mathbf{R} & h_{\kappa_1, +\infty}(t) &= t + (\kappa_1 - t)^{-1}. \end{aligned}$$

**(2.4) Definition.** Let  $\kappa_1, \kappa_2$  satisfy  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ . A curve  $\gamma: [0, 1] \rightarrow$

<sup>2</sup>The definitions given here are straightforward adaptations of the ones in [13], where they are used to study spaces of locally convex curves in  $\mathbf{S}^n$  (which correspond to the spaces  $\mathcal{L}_0^{+\infty}(Q)$  when  $n = 2$ ).



$\mathbf{S}^2$  will be called  $(\kappa_1, \kappa_2)$ -admissible if there exist  $Q_0 \in \mathbf{SO}_3$  and a pair  $(\hat{v}, \hat{w}) \in \mathbf{E}$  such that  $\gamma(t) = \Phi(t)e_1$  for all  $t \in [0, 1]$ , where  $\Phi$  is the unique solution to equation (4), with  $v, w$  given by

$$v(t) = h^{-1}(\hat{v}(t)), \quad w(t) = v(t)h_{\kappa_1, \kappa_2}^{-1}(\hat{w}(t)). \quad (6)$$

When it is not important to keep track of the bounds  $\kappa_1, \kappa_2$ , we shall say more simply that  $\gamma$  is *admissible*.

In vague but more suggestive language, an admissible curve  $\gamma$  is essentially an  $H^1$  frame  $\Phi: [0, 1] \rightarrow \mathbf{SO}_3$  such that  $\gamma = \Phi e_1: [0, 1] \rightarrow \mathbf{S}^2$  has geodesic curvature in the interval  $(\kappa_1, \kappa_2)$ . The unit tangent (resp. normal) vector  $\mathbf{t}(t) = \Phi(t)e_2$  (resp.  $\mathbf{n}(t) = \Phi(t)e_3$ ) of  $\gamma$  is thus defined everywhere on  $[0, 1]$ , and it is absolutely continuous as a function of  $t$ . The curve  $\gamma$  itself is, like  $\Phi$ , of class  $H^1$ . However, the coordinates of its velocity vector  $\dot{\gamma}(t) = v(t)\Phi(t)e_2$  lie in  $L^2[0, 1]$ , so the latter is only defined almost everywhere. The geodesic curvature of  $\gamma$ , which is also defined a.e., is given by

$$\kappa(t) = \frac{1}{v(t)} \langle \dot{\mathbf{t}}(t), \mathbf{n}(t) \rangle = h_{\kappa_1, \kappa_2}^{-1}(\hat{w}(t)) \in (\kappa_1, \kappa_2)$$

(cf. (2), (3) and (6)).

*Remark.* The reason for the choice of the specific diffeomorphism  $h: (0, +\infty) \rightarrow \mathbf{R}$  in (5) (instead of, say,  $h(t) = \log t$ ) is that we need  $h^{-1}(t)$  to diverge linearly to  $\pm\infty$  as  $t \rightarrow 0, +\infty$  in order to guarantee that  $v = h^{-1} \circ \hat{v} \in L^2[0, 1]$  whenever  $\hat{v} \in L^2[0, 1]$ . The reason for the choice of the other diffeomorphisms is analogous.

**(2.5) Definition.** Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ ,  $Q_0 \in \mathbf{SO}_3$ . Define  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0, \cdot)$  to be the set of all  $(\kappa_1, \kappa_2)$ -admissible curves  $\gamma$  such that

$$\Phi_\gamma(0) = Q_0,$$

where  $\Phi_\gamma$  is the frame of  $\gamma$ . This set is identified with  $\mathbf{E}$  via the correspondence  $\gamma \leftrightarrow (\hat{v}, \hat{w})$ , and this defines a (trivial) Hilbert manifold structure on  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0, \cdot)$ .

In particular, this space is contractible by definition. We are now ready to define the spaces  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ , which constitute the main object of study of this work.

**(2.6) Definition.** Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ ,  $Q \in \mathbf{SO}_3$ . We define  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  to be the subspace of  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I, \cdot)$  consisting of all curves  $\gamma$  in the latter space satisfying

$$\Phi_\gamma(0) = I \quad \text{and} \quad \Phi_\gamma(1) = Q. \quad (\text{i})$$

Here  $\Phi_\gamma$  is the frame of  $\gamma$  and  $I$  is the  $3 \times 3$  identity matrix.<sup>3</sup>

Because  $\mathbf{SO}_3$  has dimension 3, the condition  $\Phi_\gamma(1) = Q$  implies that  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  is a closed submanifold of codimension 3 in  $\mathbf{E} \equiv \mathcal{L}_{\kappa_1}^{\kappa_2}(I, \cdot)$ . (Here we are using the fact that the map which sends the pair  $(\hat{v}, \hat{w}) \in \mathbf{E}$  to  $\Phi(1)$  is a submersion; a proof of this when  $\kappa_1 = 0$  and  $\kappa_2 = +\infty$  can be found in §3 of [12], and the proof of the general case is analogous.) The space  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  consists of closed curves only when  $Q = I$ . Also, when  $\kappa_1 = -\infty$  and  $\kappa_2 = +\infty$  simultaneously, no restrictions are placed on the geodesic curvature. The resulting space (for arbitrary  $Q \in \mathbf{SO}_3$ ) is known to be homotopy equivalent to  $\Omega\mathbf{S}^3 \sqcup \Omega\mathbf{S}^3$ ; see the discussion after (2.13).

Note that we have natural inclusions  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \hookrightarrow \mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(Q)$  whenever  $\bar{\kappa}_1 \leq \kappa_1 < \kappa_2 \leq \bar{\kappa}_2$ . More explicitly, this map is given by:

$$\gamma \equiv (\hat{v}, \hat{w}) \mapsto (\hat{v}, h_{\bar{\kappa}_1, \bar{\kappa}_2} \circ h_{\kappa_1, \kappa_2}^{-1}(\hat{w}));$$

it is easy to check that the actual curve associated with the pair of functions in  $\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(Q)$  on the right side (via (3), (4) and (6)) is the original curve  $\gamma$ , so that the use of the term “inclusion” is justified. In fact, this map is an embedding, so that  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  can be considered a subspace of  $\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(Q)$  when  $\bar{\kappa}_1 \leq \kappa_1 < \kappa_2 \leq \bar{\kappa}_2$ .

The next lemma contains all results on Hilbert manifolds that we shall use.

**(2.7) Lemma.** *Let  $\mathcal{M}$  be a Hilbert manifold. Then:*

- (a)  *$\mathcal{M}$  is locally path-connected. In particular, its connected components and path components coincide.*
- (b) *If  $\mathcal{M}$  is weakly contractible then it is contractible.<sup>4</sup>*
- (c) *Assume that 0 is a regular value of  $F: \mathcal{M} \rightarrow \mathbf{R}^n$ . Then  $\mathcal{P} = F^{-1}(0)$  is a closed submanifold which has codimension  $n$  and trivial normal bundle in  $\mathcal{M}$ .*
- (d) *Let  $\mathbf{E}$  and  $\mathbf{F}$  be separable Banach spaces. Suppose  $i: \mathbf{F} \rightarrow \mathbf{E}$  is a bounded, injective linear map with dense image and  $M \subset \mathbf{E}$  is a smooth closed submanifold of finite codimension. Then  $N = i^{-1}(M)$  is a smooth closed submanifold of  $\mathbf{F}$  and  $i: (\mathbf{F}, N) \rightarrow (\mathbf{E}, M)$  is a homotopy equivalence of pairs.*

<sup>3</sup>The letter ‘L’ in  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  is a reference to John A. Little, who determined the connected components of  $\mathcal{L}_0^{+\infty}(I)$  in [8].

<sup>4</sup>Recall that a map  $f: X \rightarrow Y$  between topological spaces  $X$  and  $Y$  is said to be a weak homotopy equivalence if  $f_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$  is an isomorphism for any  $n \geq 0$  and  $x_0 \in X$ . The space  $X$  is said to be weakly contractible if it is weakly homotopy equivalent to a point, that is, if all of its homotopy groups are trivial.

*Proof.* Part (a) is obvious and part (b) is a special case of thm. 15 in [9]. The first assertion of part (c) is a consequence of the implicit function theorem (for Banach spaces). The triviality of the normal bundle can be proved as follows: Let  $p \in \mathcal{P}$  and  $N\mathcal{P}_p$  be the fiber over  $p$  of the normal bundle  $N\mathcal{P}$ . Then

$$T\mathcal{M}_p = T\mathcal{P}_p \oplus N\mathcal{P}_p,$$

and  $T\mathcal{P}_p$  lies in the kernel of the derivative  $TF_p$  by hypothesis, as  $F$  vanishes identically on  $\mathcal{P}$ . Since  $TF_p$  is surjective and  $\dim N\mathcal{P}_p = n$ ,  $TF_p$  must be an isomorphism when restricted to  $N\mathcal{P}_p$ . This is valid for any  $p \in \mathcal{P}$ , so we can obtain a trivialization  $\tau$  of  $N\mathcal{P}$  by setting:

$$\tau(p, v) = ((TF_p)|_{N\mathcal{P}_p})^{-1}(v) \quad (p \in \mathcal{P}, v \in \mathbf{R}^n).$$

Finally, part (d) is thm. 2 in [2]. □

**(2.8) Lemma.** *Let  $r \in \{2, 3, \dots, \infty\}$ . Then the subset of all  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  of class  $C^r$  is dense in  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ .*

*Proof.* This follows from the fact that the set of smooth functions  $f: [0, 1] \rightarrow \mathbf{R}$  is dense in  $L^2[0, 1]$ . □

**(2.9) Definition.** Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ ,  $Q \in \mathbf{SO}_3$  and  $r \in \mathbf{N}$ ,  $r \geq 2$ . Define  $\mathcal{C}_{\kappa_1}^{\kappa_2}(Q)$  to be the set, furnished with the  $C^r$  topology, of all  $C^r$  regular curves  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  such that:

- (i)  $\Phi_\gamma(0) = I$  and  $\Phi_\gamma(1) = Q$ ;
- (ii)  $\kappa_1 < \kappa(t) < \kappa_2$  for each  $t \in [0, 1]$ .

The value of  $r$  is not important, as all of these spaces are homotopy equivalent. Because of this, after the next lemma, when we speak of  $\mathcal{C}_{\kappa_1}^{\kappa_2}(Q)$ , we will implicitly assume that  $r = 2$ .

**(2.10) Lemma.** *Let  $r \in \mathbf{N}$  ( $r \geq 2$ ),  $Q \in \mathbf{SO}_3$  and  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ . Then the set inclusion  $i: \mathcal{C}_{\kappa_1}^{\kappa_2}(Q) \hookrightarrow \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  is a homotopy equivalence.*

*Proof.* In this proof we will highlight the differentiability class by denoting  $\mathcal{C}_{\kappa_1}^{\kappa_2}(Q)$  by  $\mathcal{C}_{\kappa_1}^{\kappa_2}(Q)^r$ . Let  $\mathbf{E} = L^2[0, 1] \times L^2[0, 1]$ , let  $\mathbf{F} = C^{r-1}[0, 1] \times C^{r-2}[0, 1]$  (where  $C^k[0, 1]$  denotes the set of all  $C^k$  functions  $[0, 1] \rightarrow \mathbf{R}$ , with the  $C^k$  norm) and let  $i: \mathbf{E} \rightarrow \mathbf{F}$  be set inclusion. Setting  $M = \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ , we conclude from (2.7(d)) that  $i: N = i^{-1}(M) \hookrightarrow M$  is a homotopy equivalence. We claim that  $N \approx \mathcal{C}_{\kappa_1}^{\kappa_2}(Q)^r$ , where the homeomorphism is obtained by associating a

pair  $(\hat{v}, \hat{w}) \in N$  to the curve  $\gamma$  obtained by solving (4) (with  $\Lambda$  defined by (3) and (6) and  $Q_0 = I$ ), and vice-versa.

Suppose first that  $\gamma \in \mathcal{C}_{\kappa_1}^{\kappa_2}(Q)^r$ . Then  $|\dot{\gamma}|$  (resp.  $\kappa$ ) is a function  $[0, 1] \rightarrow \mathbf{R}$  of class  $C^{r-1}$  (resp.  $C^{r-2}$ ). Hence, so are  $\hat{v} = h \circ |\dot{\gamma}|$  and  $\hat{w} = h_{\kappa_1}^{\kappa_2} \circ \kappa$ , since  $h$  and  $h_{\kappa_1}^{\kappa_2}$  are smooth. Conversely, if  $(\hat{v}, \hat{w}) \in N$ , then  $v = h^{-1}(\hat{v})$  is of class  $C^{r-1}$  and  $w = (h_{\kappa_1}^{\kappa_2})^{-1} \circ \hat{w}$  of class  $C^{r-2}$ , and the frame  $\Phi$  of the curve  $\gamma$  corresponding to that pair satisfies

$$\dot{\Phi} = \Phi \Lambda, \quad \Lambda = \begin{pmatrix} 0 & -|\dot{\gamma}| & 0 \\ |\dot{\gamma}| & 0 & -|\dot{\gamma}| \kappa \\ 0 & |\dot{\gamma}| \kappa & 0 \end{pmatrix} = \begin{pmatrix} 0 & -v & 0 \\ v & 0 & -w \\ 0 & w & 0 \end{pmatrix}.$$

Since the entries of  $\Lambda$  are of class (at least)  $C^{r-2}$ , the entries of  $\Phi$  are functions of class  $C^{r-1}$ . Moreover,  $\gamma = \Phi e_1$ , hence

$$\dot{\gamma} = \dot{\Phi} e_1 = \Phi \Lambda e_1 = v \Phi e_2,$$

and the velocity vector of  $\gamma$  is seen to be of class  $C^{r-1}$ . It follows that  $\gamma$  is a curve of class  $C^r$ . Finally, it is easy to check that the correspondence  $(\hat{v}, \hat{w}) \leftrightarrow \gamma$  is continuous in both directions.  $\square$

### Lifted frames

The (two-sheeted) universal covering space of  $\mathbf{SO}_3$  is  $\mathbf{S}^3$ . Let us briefly recall the definition of the covering map  $\pi: \mathbf{S}^3 \rightarrow \mathbf{SO}_3$ .<sup>5</sup> We start by identifying  $\mathbf{R}^4$  with the algebra  $\mathbf{H}$  of quaternions, and  $\mathbf{S}^3$  with the subgroup of unit quaternions. Given  $z \in \mathbf{S}^3$ ,  $v \in \mathbf{R}^4$ , define a transformation  $T_z: \mathbf{R}^4 \rightarrow \mathbf{R}^4$  by  $T_z(v) = zvz^{-1} = zv\bar{z}$ . One checks easily that  $T_z$  preserves the sum, multiplication and conjugation operations. It follows that, for any  $v, w \in \mathbf{R}^4$ ,

$$\begin{aligned} 4 \langle T_z(v), T_z(w) \rangle &= |T_z(v) + T_z(w)|^2 - |T_z(v) - T_z(w)|^2 \\ &= |v + w|^2 - |v - w|^2 = 4 \langle v, w \rangle, \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  denotes the usual inner product in  $\mathbf{R}^4$ . Thus  $T_z$  is an orthogonal linear transformation of  $\mathbf{R}^4$ . Moreover,  $T_z(\mathbf{1}) = \mathbf{1}$  (where  $\mathbf{1}$  is the unit of  $\mathbf{H}$ ), hence the three-dimensional vector subspace  $\{0\} \times \mathbf{R}^3 \subset \mathbf{R}^4$  consisting of the purely imaginary quaternions is invariant under  $T_z$ . The element  $\pi(z) \in \mathbf{SO}_3$  is the restriction of  $T_z$  to this subspace, where  $(a, b, c) \in \mathbf{R}^3$  is identified with the quaternion  $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ .

<sup>5</sup>See [4] for more details and further information on quaternions and rotations.

In what follows we adopt the convention that  $\mathbf{S}^3$  (resp.  $\mathbf{SO}_3$ ) is furnished with the Riemannian metric inherited from  $\mathbf{R}^4$  (resp.  $\mathbf{R}^9$ ).

**(2.11) Lemma.** *Let  $\langle \cdot, \cdot \rangle$  denote the metric in  $\mathbf{S}^3$  and  $\langle \cdot, \cdot \rangle$  the metric in  $\mathbf{SO}_3$ . Then  $\pi^*\langle \cdot, \cdot \rangle = 8\langle \cdot, \cdot \rangle$ , where  $\pi^*\langle \cdot, \cdot \rangle$  denotes the pull-back of  $\langle \cdot, \cdot \rangle$  by  $\pi$ .*

*Proof.* It suffices to prove that if

$$z: (-1, 1) \rightarrow \mathbf{S}^3, \quad t \mapsto a(t)\mathbf{1} + b(t)\mathbf{i} + c(t)\mathbf{j} + d(t)\mathbf{k}$$

is a regular curve and  $Q = \pi \circ z$  then  $|\dot{Q}(0)|^2 = 8|\dot{z}(0)|^2$ . Let us assume first that  $z(0) = \mathbf{1}$ , so that  $\dot{a}(0) = 0$ . From the definition of  $Q$ , we have

$$Q(t)e_1 = z(t)\mathbf{i}\bar{z}(t)$$

and similarly for  $\mathbf{j}, \mathbf{k}$ , where, as above, we identify  $\mathbf{R}^3$  with the imaginary quaternions. Hence

$$\begin{aligned} |\dot{Q}(0)e_1|^2 &= |z(0)\mathbf{i}\dot{z}(0) + \dot{z}(0)\mathbf{i}\bar{z}(0)|^2 = 2|\dot{z}(0)|^2 - (\dot{z}(0)\mathbf{i})^2 - (\mathbf{i}\dot{z}(0))^2 \\ &= 2|\dot{z}(0)|^2 - 2\operatorname{Re}((\dot{z}(0)\mathbf{i})^2) \end{aligned}$$

Therefore

$$|\dot{Q}(0)|^2 = 6|\dot{z}(0)|^2 - 2\operatorname{Re}((\dot{z}(0)\mathbf{i})^2) - 2\operatorname{Re}((\dot{z}(0)\mathbf{j})^2) - 2\operatorname{Re}((\dot{z}(0)\mathbf{k})^2)$$

Since  $\operatorname{Re}(w^2) = \alpha^2 - \beta^2 - \gamma^2 - \delta^2$  if  $w = \alpha + \beta\mathbf{i} + \gamma\mathbf{j} + \delta\mathbf{k}$  and  $\dot{a}(0) = 0$ , we deduce that

$$-2\operatorname{Re}((\dot{z}(0)\mathbf{i})^2) = 2\dot{c}(0)^2 + 2\dot{d}(0)^2 - 2\dot{b}(0)^2 = 2|\dot{z}(0)|^2 - 4\dot{b}(0)^2$$

and analogously for  $\mathbf{j}, \mathbf{k}$ . Thus  $|\dot{Q}(0)|^2 = 8|\dot{z}(0)|^2$  as claimed, provided  $z(0) = \mathbf{1}$ .

Now consider any regular curve  $w: (-1, 1) \rightarrow \mathbf{S}^3$ , let  $P = \pi \circ w$  and set

$$z(t) = w(0)^{-1}w(t), \quad Q(t) = \pi(z(t)) = P(0)^{-1}P(t).$$

Then  $z(0) = \mathbf{1}$ , hence

$$|\dot{P}(0)|^2 = |P(0)\dot{Q}(0)|^2 = |\dot{Q}(0)|^2 = 8|\dot{z}(0)|^2 = 8|w(0)\dot{z}(0)|^2 = 8|\dot{w}(0)|^2. \quad \square$$

**(2.12) Definition.** Let  $\Phi: [0, 1] \rightarrow \mathbf{SO}_3$  be a frame (of class  $H^1$ ) and let  $z \in \mathbf{S}^3$  satisfy  $\pi(z) = \Phi_\gamma(0)$ . We define the *lifted frame*  $\tilde{\Phi}^z: [0, 1] \rightarrow \mathbf{S}^3$  to be the lift

of  $\Phi$  to  $\mathbf{S}^3$ , starting at  $z$ . When  $\Phi(0) = I$  we adopt the convention that  $z = \mathbf{1}$ , and we denote the lifted frame simply by  $\tilde{\Phi}$ .

Here is a simple but important application of this concept.

**(2.13) Lemma.** *Let  $\gamma_0, \gamma_1 \in \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ , for some  $Q \in \mathbf{SO}_3$ , and suppose that  $\gamma_0, \gamma_1$  lie in the same connected component of this space. Then  $\tilde{\Phi}_{\gamma_0}(1) = \tilde{\Phi}_{\gamma_1}(1)$ .*

*Proof.* Since  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  is a Hilbert manifold, its path and connected components coincide. Therefore, to say that  $\gamma_0, \gamma_1$  lie in the same connected component of  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  is the same as to say that there exists a continuous family of curves  $\gamma_s \in \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  joining  $\gamma_0$  and  $\gamma_1$ ,  $s \in [0, 1]$ . The family  $\Phi_{\gamma_s}$  yields a homotopy between the paths  $\Phi_{\gamma_0}$  and  $\Phi_{\gamma_1}$  in  $\mathbf{SO}_3$ . (Recall that each of the frames  $\Phi_{\gamma_s}$  is (absolutely) continuous.) By the homotopy lifting property of covering spaces, the paths  $\tilde{\Phi}_{\gamma_0}$  and  $\tilde{\Phi}_{\gamma_1}$  are also homotopic in  $\mathbf{S}^3$  (fixing the endpoints).  $\square$

### The role of the initial and final frames

We will now study how the topology of  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  changes if we consider variations of condition (i) in (2.6); by the end of the section it should be clear that our original definition is sufficiently general. A summary of all the definitions considered here is given in table form on p. 28.

For fixed  $z \in \mathbf{S}^3$ , let  $\Omega_z \mathbf{S}^3$  denote the set of all continuous paths  $\omega: [0, 1] \rightarrow \mathbf{S}^3$  such that  $\omega(0) = \mathbf{1}$  and  $\omega(1) = z$ , furnished with the compact-open topology. It can be shown (see [1], p. 198) that  $\Omega_z \mathbf{S}^3 \simeq \Omega \mathbf{S}^3$  for any  $z \in \mathbf{S}^3$ , where  $\Omega \mathbf{S}^3$  is the space of paths in  $\mathbf{S}^3$  which start and end at  $\mathbf{1} \in \mathbf{S}^3$ .<sup>6</sup> The topology of this space is well understood; we refer the reader to [1], §16, for more information.

Now let  $\kappa_1 < \kappa_2$ ,  $z \in \mathbf{S}^3$  be arbitrary and  $Q = \pi(z)$ . Define

$$F: \mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \rightarrow \Omega_z \mathbf{S}^3 \cup \Omega_{-z} \mathbf{S}^3 \simeq \Omega \mathbf{S}^3 \sqcup \Omega \mathbf{S}^3 \quad \text{by} \quad F(\gamma) = \tilde{\Phi}_{\gamma}. \quad (7)$$

In the special case  $\kappa_1 = -\infty$ ,  $\kappa_2 = +\infty$ , it follows from the Hirsch-Smale theorem that this map is a homotopy equivalence. In the general case this is false, however. For instance,  $\Omega \mathbf{S}^3 \sqcup \Omega \mathbf{S}^3$  has two connected components, while Little has proved ([8], thm. 1) that  $\mathcal{L}_0^{+\infty}(I)$  has three connected components. We take this opportunity to recall the precise statement of Little's theorem and to introduce a new class of spaces.

<sup>6</sup>The notation  $X \simeq Y$  (resp.  $X \approx Y$ ) means that  $X$  is homotopy equivalent (resp. homeomorphic) to  $Y$ .

**(2.14) Definition.** Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ . Define  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  to be the space of all  $(\kappa_1, \kappa_2)$ -admissible curves  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  such that

$$\Phi_\gamma(0) = \Phi_\gamma(1).$$

Note that the only difference between  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  and  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  is that curves in the latter space may have arbitrary initial and final frames, as long as they coincide. An argument analogous to the one given for the spaces  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  shows that  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  is also a Hilbert manifold. In fact, we have the following relationship between the two classes.

**(2.15) Proposition.** *The space  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  is homeomorphic to  $\mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ .*

*Proof.* For  $Q \in \mathbf{SO}_3$  and  $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ , let  $Q\gamma$  be the curve defined by  $(Q\gamma)(t) = Q(\gamma(t))$ . Because  $Q$  is an isometry, the geodesic curvatures of  $Q\gamma$  at  $(Q\gamma)(t)$  and of  $\gamma$  at  $\gamma(t)$  coincide. Define  $F: \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I) \rightarrow \mathcal{L}_{\kappa_1}^{\kappa_2}$  by  $F(Q, \gamma) = Q\gamma$ ; clearly,  $F$  is continuous. Since it has the continuous inverse  $\eta \mapsto (\Phi_\eta(0), \Phi_\eta(0)^{-1}\eta)$ ,  $F$  is a homeomorphism.  $\square$

Let us temporarily denote by  $\mathcal{L}$  the space  $\mathcal{L}_{-\infty}^0 \sqcup \mathcal{L}_0^{+\infty}$  studied by Little. We have  $\mathcal{L}_{-\infty}^0 \approx \mathcal{L}_0^{+\infty}$ , since the map which takes a curve in  $\mathcal{L}$  to the same curve with reversed orientation is a (self-inverse) homeomorphism mapping  $\mathcal{L}_{-\infty}^0$  onto  $\mathcal{L}_0^{+\infty}$ . What is proved in [8] is that  $\mathcal{L}$  has six connected components.<sup>7</sup> Using prop. (2.15) and the fact that  $\mathbf{SO}_3$  is connected, we see that Little's theorem is equivalent to the assertion that  $\mathcal{L}_0^{+\infty}(I)$  has three connected components, as was claimed immediately above (2.14).

A natural generalization of the spaces  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  is obtained by modifying condition (i) of (2.6) as follows.

**(2.16) Definition.** Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$  and  $Q_0, Q_1 \in \mathbf{SO}_3$ . Define  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0, Q_1)$  to be the space of all  $(\kappa_1, \kappa_2)$ -admissible curves  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  such that

$$\Phi_\gamma(0) = Q_0 \quad \text{and} \quad \Phi_\gamma(1) = Q_1. \quad (\text{i}')$$

Thus, the only difference between condition (i) on p. 17 and condition (i') is that the latter allows arbitrary initial frames.

**(2.17) Proposition.**  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0, Q_1) \approx \mathcal{L}_{\kappa_1}^{\kappa_2}(PQ_0, PQ_1)$  for any  $P, Q_0, Q_1 \in \mathbf{SO}_3$ . Then. In particular,  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0, Q_1) \approx \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ , where  $Q = Q_0^{-1}Q_1$ .

*Proof.* The proof is similar to that of (2.15). The map  $\gamma \mapsto P\gamma$  takes  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0, Q_1)$  into  $\mathcal{L}_{\kappa_1}^{\kappa_2}(PQ_0, PQ_1)$  and is continuous. The map  $\gamma \mapsto P^{-1}\gamma$ , which is likewise continuous, is its inverse.  $\square$

<sup>7</sup>Little works with  $C^2$  curves, but, as we have seen, this is not important.

Of course, we could also consider the spaces  $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, Q)$ , consisting of all  $(\kappa_1, \kappa_2)$ -admissible curves  $\gamma$  having final frame  $\Phi_\gamma(1) = Q \in \mathbf{SO}_3$  (but arbitrary initial frame). Like  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q, \cdot)$ , this space is contractible. To see this, one can go through the definition to check that it is indeed diffeomorphic to  $\mathbf{E}$ , or, alternatively, one can observe that the map  $\gamma \mapsto \bar{\gamma}$ ,  $\bar{\gamma}(t) = \gamma(1-t)$ , establishes a homeomorphism

$$\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, Q) \approx \mathcal{L}_{\kappa_1}^{\kappa_2}(QR, \cdot),$$

where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Finally, we could study the space  $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$  of all  $(\kappa_1, \kappa_2)$ -admissible curves, with no conditions placed on the frames. The argument given in the proof of (2.15) shows that

$$\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, \cdot) \approx \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I, \cdot).$$

Hence,  $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$  is homeomorphic to  $\mathbf{E} \times \mathbf{SO}_3$ , and has the homotopy type of  $\mathbf{SO}_3$ .

Thus, the topology of the spaces  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q, \cdot)$ ,  $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, Q)$  and  $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$  is uninteresting. We will have nothing else to say about these spaces.

### The role of the bounds on the curvature

Having analyzed the significance of condition (i) on p. 14, let us examine next condition (ii). Notice that we have allowed the bounds  $\kappa_1, \kappa_2$  on the curvature to be infinite. The definition of radius of curvature is extended accordingly by setting  $\operatorname{arccot}(+\infty) = 0$  and  $\operatorname{arccot}(-\infty) = \pi$ . We can then rephrase (ii) as:

$$(ii) \quad \rho(t) \in (\rho_2, \rho_1) \text{ for each } t \in [0, 1].$$

Here  $\rho$  is the radius of curvature of  $\gamma$  and  $\rho_i = \operatorname{arccot} \kappa_i \in [0, \pi]$ ,  $i = 1, 2$ . The main result of this section relates the topology of  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  to the size  $\rho_1 - \rho_2$  of the interval  $(\rho_2, \rho_1)$ . Its proof relies on the following construction.

Given  $-\pi < \theta < \pi$  and an admissible curve  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$ , define the *translation*  $\gamma_\theta: [0, 1] \rightarrow \mathbf{S}^2$  of  $\gamma$  by  $\theta$  to be the curve given by

$$\gamma_\theta(t) = \cos \theta \gamma(t) + \sin \theta \mathbf{n}(t) \quad (t \in [0, 1]). \quad (8)$$

*Example.* Let  $0 < \alpha < \frac{\pi}{2}$  and let  $C$  be the circle of colatitude  $\alpha$ . Depending on the orientation, the translation of  $C$  by  $\theta$ ,  $0 \leq \theta \leq \alpha$ , is either the circle of



colatitude  $\alpha + \theta$  or the circle of colatitude  $\alpha - \theta$ . In particular, taking  $\theta = \alpha$  and a suitable orientation of  $C$ , the translation degenerates to a single point (the north pole).

This example shows that some care must be taken in the choice of  $\theta$  for the resulting curve to be admissible.

**(2.18) Lemma.** *Let  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  be an admissible curve and  $\rho$  its radius of curvature. Suppose*

$$\rho_2 < \rho(t) < \rho_1 \text{ for a.e. } t \in [0, 1] \text{ and } \rho_1 - \pi \leq \theta \leq \rho_2. \quad (9)$$

*Then  $\gamma_\theta$  is an admissible curve and its frame is given by:*

$$\Phi_{\gamma_\theta} = \Phi_\gamma R_\theta, \quad \text{where} \quad R_\theta = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}. \quad (10)$$

*Proof.* Let  $\Psi = \Phi_\gamma R_\theta$ . Since  $\Phi_\gamma$  satisfies the differential equation (2),  $\Psi$  satisfies

$$\dot{\Psi} = \Psi (R_\theta^{-1} \Lambda R_\theta).$$

A direct calculation shows that

$$R_\theta^{-1} \Lambda R_\theta = \begin{pmatrix} 0 & -(\cos \theta v - \sin \theta w) & 0 \\ \cos \theta v - \sin \theta w & 0 & -(\cos \theta w + \sin \theta v) \\ 0 & \cos \theta w + \sin \theta v & 0 \end{pmatrix},$$

where  $v = v(t) = |\dot{\gamma}(t)|$  and  $w = w(t) = v(t)\kappa(t)$ . Also,  $\Psi e_1 = \gamma_\theta$  by construction. To show that  $\gamma_\theta$  is admissible, it is thus only necessary to show that

$$\cos \theta v(t) - \sin \theta w(t) = v(t)(\cos \theta - \sin \theta \cot \rho(t)) = \frac{v(t)}{\sin \rho(t)} \sin(\rho(t) - \theta) > 0$$

for almost every  $t \in [0, 1]$ , and this is true by our choice of  $\theta$  and the fact that  $v > 0$ .  $\square$

Thus, for  $\theta$  satisfying (9), we obtain from (10) that the unit tangent vector  $\mathbf{t}_\theta$  and unit normal vector  $\mathbf{n}_\theta$  to the translation  $\gamma_\theta$  of  $\gamma$  are given by:

$$\mathbf{t}_\theta(t) = \mathbf{t}(t) \quad \text{and} \quad \mathbf{n}_\theta(t) = -\sin \theta \gamma(t) + \cos \theta \mathbf{n}(t) \quad (11)$$

for almost every  $t \in [0, 1]$ .

**(2.19) Lemma.** *Let  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  be an admissible curve and suppose that (9) holds. Then  $(\gamma_\theta)_\varphi = \gamma_{\theta+\varphi}$  for any  $\varphi \in (-\pi, \pi)$ . In particular,  $(\gamma_\theta)_{-\theta} = \gamma$ .*

*Proof.* Note that  $(\gamma_\theta)_\varphi$  is defined because  $\gamma_\theta$  is admissible, as we have just seen. Using (8) and (11) we obtain that

$$(\gamma_\theta)_\varphi = \cos \varphi (\cos \theta \gamma + \sin \theta \mathbf{n}) + \sin \varphi (-\sin \theta \gamma + \cos \theta \mathbf{n}) = \gamma_{\theta+\varphi}. \quad \square$$

Given three distinct points on  $\mathbf{S}^2$ , there is a unique circle passing through them; this circle is also contained in the sphere, for it is the intersection of the unique plane containing the points with  $\mathbf{S}^2$ . Now consider a  $C^2$  regular curve  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$ . Fix  $t \in [0, 1]$ , and take distinct  $t_1, t_2, t_3 \in [0, 1]$ . Because the Euclidean curvature  $K(t) \neq 0$ , the osculating circle to  $\gamma$  at  $\gamma(t)$  exists and is equal to the limit position, as  $t_1, t_2, t_3$  approach  $t$ , of the unique circle through  $\gamma(t_1), \gamma(t_2)$  and  $\gamma(t_3)$ . Therefore, being a limit of circles contained in the sphere, the osculating circle at any point of  $\gamma$  is also contained in the sphere.

**(2.20) Lemma.** *Let  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  be  $C^2$  regular and let  $\theta$  satisfy (9). Then the osculating circle to the translation  $\gamma_\theta$  at  $\gamma_\theta(t)$  is the translation of the osculating circle to  $\gamma$  at  $\gamma(t)$  by  $\theta$ .*

*Proof.* Let  $\gamma$  be parametrized by arc-length and let  $\sigma$  be a parametrization, also by arc-length, of the osculating circle to  $\gamma$  at  $\gamma(0)$ . By definition, the osculating circle is the unique circle in  $\mathbf{R}^3$  which has contact of order 3 with  $\gamma$  at  $\gamma(0)$ ; that is,  $\sigma$  must satisfy:

$$\sigma(0) = \gamma(0), \quad \sigma'(0) = \gamma'(0), \quad \sigma''(0) = \gamma''(0).$$

In particular, the geodesic curvatures of  $\gamma$  and  $\sigma$  at the point  $\gamma(0) = \sigma(0)$  coincide. From these relations and (8) we deduce that  $\sigma_\theta(0) = \gamma_\theta(0)$ ,  $\dot{\sigma}_\theta(0) = \dot{\gamma}_\theta(0)$ . Another calculation shows that

$$\begin{aligned} \ddot{\gamma}_\theta(0) &= (\kappa(0) \sin \theta - \cos \theta) (\gamma(0) - \kappa(0) \mathbf{n}(0)) - \kappa'(0) \sin \theta \mathbf{t}(0), \\ \ddot{\sigma}_\theta(0) &= (\kappa(0) \sin \theta - \cos \theta) (\sigma(0) - \kappa(0) \mathbf{n}(0)). \end{aligned}$$

(Here  $\gamma_\theta$  (resp.  $\sigma_\theta$ ) is parametrized with respect to the arc-length parameter of  $\gamma$  (resp.  $\sigma$ .) This shows that the vector subspaces of  $\mathbf{R}^3$  spanned by the two pairs  $\{\dot{\gamma}_\theta(0), \ddot{\gamma}_\theta(0)\}$  and  $\{\dot{\sigma}_\theta(0), \ddot{\sigma}_\theta(0)\}$  coincide. Consequently, the image of  $\sigma_\theta$  is a circle in the sphere contained in the plane parallel to  $\dot{\gamma}_\theta(0)$  and  $\ddot{\gamma}_\theta(0)$  through  $\gamma_\theta(0)$ . But there is only one such circle, viz., the osculating circle to  $\gamma_\theta$  at  $\gamma_\theta(0)$ . Since 0 could have been replaced by any  $s_0 \in [0, 1]$  in this argument, the proof is complete.  $\square$

**(2.21) Corollary.** *Let  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  be an admissible curve and let  $\theta$  satisfy (9). Then the radius of curvature  $\bar{\rho}$  of  $\gamma_\theta$  is given by  $\bar{\rho} = \rho - \theta$ .*

*Proof.* If  $\gamma$  is  $C^2$  regular we can, by (2.20), actually assume that it is a circle. Then an easy direct verification shows that the formula  $\bar{\rho} = \rho - \theta$  holds regardless of which orientation we choose. The general case where  $\gamma$  is only admissible can be deduced from this by applying (2.8).  $\square$

**(2.22) Theorem.** *Let  $Q \in \mathbf{SO}_3$ ,  $\kappa_1 < \kappa_2$ ,  $\bar{\kappa}_1 < \bar{\kappa}_2$ ,  $\rho_i = \operatorname{arccot} \kappa_i$ ,  $\bar{\rho}_i = \operatorname{arccot} \bar{\kappa}_i$ . Suppose that  $\rho_1 - \rho_2 = \bar{\rho}_1 - \bar{\rho}_2$ . Then  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \approx \mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(R_{-\theta}QR_\theta)$ , where  $\theta = \rho_2 - \bar{\rho}_2$  and*

$$R_\theta = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}.$$

We recall that the bounds  $\kappa_i$ ,  $\bar{\kappa}_i$  may take on infinite values, and we adopt the conventions that  $\operatorname{arccot}(+\infty) = 0$  and  $\operatorname{arccot}(-\infty) = \pi$ .

*Proof.* Let  $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  and let  $\rho$  be its radius of curvature. We have:

$$\rho_2 < \rho(t) < \rho_1 \text{ for a.e. } t \in [0, 1].$$

Set  $\theta = \rho_2 - \bar{\rho}_2$ . Then (9) is satisfied, so  $\gamma_\theta$  is an admissible curve. By (2.21), the radius of curvature  $\bar{\rho}$  of  $\gamma_\theta$  is given by  $\bar{\rho} = \rho - \theta$ . Thus,

$$\bar{\rho}_2 < \bar{\rho}(t) < \bar{\rho}_1 \text{ for a.e. } t \in [0, 1].$$

Together with (2.18), this says that  $F: \gamma \mapsto \gamma_\theta$  maps  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  into  $\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(R_\theta, QR_\theta)$ . Similarly, translation by  $-\theta$  is a map  $G: \mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(R_\theta, QR_\theta) \rightarrow \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ . By (2.19), the maps  $F$  and  $G$  are inverse to each other, hence

$$\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \approx \mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(R_\theta, QR_\theta).$$

Finally, because  $R_\theta^{-1} = R_{-\theta}$ , (2.17) guarantees that

$$\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(R_\theta, QR_\theta) \approx \mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(R_{-\theta}QR_\theta). \quad \square$$

**(2.23) Remark.** Taking  $Q = I$  we obtain from (2.22) that  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I) \approx \mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(I)$  ( $\kappa_i$ ,  $\bar{\kappa}_i$  as in the hypothesis of the theorem). It will also be important to us that under the homeomorphisms of (2.22) and the following corollaries, the image of any circle traversed  $k$  times is another circle traversed  $k$  times.

**(2.24) Corollary.** *Let  $Q \in \mathbf{SO}_3$  and  $\kappa_1 < \kappa_2$ . Then  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \approx \mathcal{L}_{-\kappa_0}^{+\kappa_0}(P)$  for suitable  $\kappa_0 > 0$ ,  $P \in \mathbf{SO}_3$ . Moreover, if  $Q = I$  then  $P = I$  also.*

*Proof.* Let  $\rho_i = \operatorname{arccot} \kappa_i$ ,  $i = 1, 2$ , and set

$$\bar{\rho}_1 = \frac{\pi}{2} + \frac{\rho_1 - \rho_2}{2}, \quad \bar{\rho}_2 = \frac{\pi}{2} - \frac{\rho_1 - \rho_2}{2} \quad \text{and} \quad \kappa_0 = \cot(\bar{\rho}_2).$$

The interval  $(\bar{\rho}_2, \bar{\rho}_1)$  has the same size as  $(\rho_2, \rho_1)$  by construction. Since  $\cot(\bar{\rho}_1) = -\kappa_0$ , (2.22) yields that  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \approx \mathcal{L}_{-\kappa_0}^{+\kappa_0}(R_{-\theta}QR_\theta)$ , where  $\theta = \frac{\rho_1 + \rho_2 - \pi}{2}$ .  $\square$

**(2.25) Corollary.** *Let  $Q \in \mathbf{SO}_3$  and  $\kappa_1 < \kappa_2$ . Then  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \approx \mathcal{L}_{\kappa_0}^{+\infty}(P)$  for suitable  $\kappa_0 \in [-\infty, +\infty)$  and  $P \in \mathbf{SO}_3$ . Moreover, if  $Q = I$  then  $P = I$  also.*

*Proof.* Let  $\rho_i = \operatorname{arccot} \kappa_i$ ,  $i = 1, 2$ . Then the interval  $(\rho_2, \rho_1)$  has the same size as the interval  $(0, \rho_1 - \rho_2)$ . Hence, by (2.22),  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \approx \mathcal{L}_{\kappa_0}^{+\infty}(R_{-\theta}QR_\theta)$ , where

$$\kappa_0 = \cot(\rho_1 - \rho_2) = \frac{1 + \kappa_1\kappa_2}{\kappa_2 - \kappa_1} \quad \text{and} \quad \theta = \rho_2. \quad \square$$

Corollaries (2.24) and (2.25) both express the fact that, for fixed  $Q \in \mathbf{SO}_3$ , the topology of the spaces  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  depends essentially on one parameter, not two. The spaces of type  $\mathcal{L}_{-\kappa_0}^{+\kappa_0}(Q)$  and  $\mathcal{L}_{\kappa_0}^{+\infty}(Q)$  have been singled out merely because they are more convenient to work with. For spaces of closed curves we have the following result relating the two classes, which is another simple consequence of (2.24).

**(2.26) Corollary.** *Let  $\kappa_0 \in [-\infty, +\infty)$ ,  $\kappa_1 \in (0, +\infty]$  and  $\rho_i = \operatorname{arccot}(\kappa_i)$ ,  $i = 0, 1$ . If  $\rho_0 = \pi - 2\rho_1$  then  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I) \approx \mathcal{L}_{\kappa_0}^{+\infty}(I)$ .*  $\square$

For convenience, we list in table 2.1 all the spaces considered thus far, together with some of the results that we have proved about their topology. As we have already remarked, the spaces  $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, Q)$ ,  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q, \cdot)$  and  $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$  will not be mentioned again.

Space	Definition	Condition on Frames	Topology
$\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$	p. 17, (2.6)	$\Phi(0) = I, \Phi(1) = Q$	depends on $\rho_1 - \rho_2, Q$
$\mathcal{L}_{\kappa_1}^{\kappa_2}$	p. 22, (2.14)	$\Phi(0) = \Phi(1)$ arbitrary	$\approx \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$
$\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0, Q_1)$	p. 23, (2.16)	$\Phi(0) = Q_0, \Phi(1) = Q_1$	$\approx \mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0^{-1}Q_1)$
$\mathcal{L}_{\kappa_1}^{\kappa_2}(Q, \cdot)$	p. 17, (2.5)	$\Phi(0) = Q, \Phi(1)$ arbitrary	contractible
$\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, Q)$	p. 24	$\Phi(0)$ arbitrary, $\Phi(1) = Q$	contractible
$\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$	p. 24	none	$\simeq \mathbf{SO}_3$

Table 2.1: Spaces of spherical curves of bounded geodesic curvature. Here  $Q \in \mathbf{SO}_3$ ,  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$  and  $\rho_i = \operatorname{arccot}(\kappa_i)$ . The notation  $X \approx Y$  (resp.  $X \simeq Y$ ) means that  $X$  is homeomorphic (resp. homotopy equivalent) to  $Y$ .

### 3

#### Curves Contained in a Hemisphere

There exists a two-way correspondence between the unit sphere  $\mathbf{S}^n$  in  $\mathbf{R}^{n+1}$  and the set consisting of its open hemispheres; namely, with  $h \in \mathbf{S}^n$  we can associate

$$H = \{p \in \mathbf{S}^n : \langle h, p \rangle > 0\}.$$

Thus the set of open hemispheres of  $\mathbf{S}^n$  carries a natural topology. For convenience, we will often identify  $H$  with  $h$ . In the sequel all hemispheres shall be open, save explicit mention to the contrary, and we will assume throughout that  $n \geq 2$ .

Let  $\gamma: [0, 1] \rightarrow \mathbf{S}^n$  be a (continuous) curve contained in the hemisphere  $H$ . As a consequence of the compactness of  $[0, 1]$ , if  $\tilde{h} \in \mathbf{S}^n$  is sufficiently close to  $h$ , then  $\gamma$  is also contained in the hemisphere  $\tilde{H}$  corresponding to  $\tilde{h}$ . It is desirable to be able to select, in a natural way, a distinguished hemisphere among those which contain  $\gamma$ .

**(3.1) Lemma.** *Let  $\gamma: [0, 1] \rightarrow \mathbf{S}^n$  be contained in a hemisphere. Then the set  $\mathcal{H} \subset \mathbf{S}^n$  of hemispheres that contain  $\gamma$  is open, geodesically convex and itself contained in a hemisphere.<sup>1</sup>*

*Proof.* The hemisphere determined by  $\gamma(0)$  contains  $\mathcal{H}$  since  $\langle h, \gamma(0) \rangle > 0$  for each  $h \in \mathcal{H}$ . Suppose that the hemispheres  $H, \tilde{H}$  corresponding respectively to  $h, \tilde{h} \in \mathbf{S}^n$  belong to  $\mathcal{H}$ . We lose no generality in assuming that

$$h = e_1, \quad \tilde{h} = e^{i\theta_0} = \cos \theta_0 e_1 + \sin \theta_0 e_2, \quad \text{where } 0 < \theta_0 < \pi.^2$$

Any  $k$  in the shortest geodesic through  $h, \tilde{h}$  has the form

$$k = e^{i\theta}, \quad \text{where } 0 \leq \theta \leq \theta_0,$$

while any  $p \in \mathbf{S}^n$  satisfying both  $\langle p, h \rangle > 0$  and  $\langle p, \tilde{h} \rangle > 0$  is of the form

$$p = e^{i\phi} + \nu, \quad \text{where } \theta_0 - \pi/2 < \phi < \pi/2 \text{ and } \nu \text{ is normal to } e_1 \text{ and } e_2.$$

<sup>1</sup>See the appendix for the definition and basic properties of geodesically convex sets.

<sup>2</sup>The use of complex numbers here is made only to simplify the notation.

The bounds on  $\theta$  and  $\phi$  give  $|\theta - \phi| < \pi/2$ , hence  $\langle p, k \rangle = \cos(\theta - \phi) > 0$ . Thus  $p \in K$  (the hemisphere determined by  $k$ ) whenever  $p \in H, \tilde{H}$ , that is,  $\mathcal{H}$  is geodesically convex. Finally, we have already remarked above that  $\mathcal{H}$  is open.  $\square$

From (3.1) we deduce that the barycenter (in  $\mathbf{R}^{n+1}$ ) of the set  $\mathcal{H}$  of hemispheres containing  $\gamma$  is not the origin. Its image under gnomonic (i.e., central) projection on the sphere, to be denoted by  $h_\gamma$ , will be our choice of distinguished hemisphere containing  $\gamma$ .

**(3.2) Lemma.** *Let  $r \geq 0$ , let  $\mathcal{A}$  denote the space of arcs  $\gamma: [0, 1] \rightarrow \mathbf{S}^n$ , with the  $C^r$  topology, and let  $\mathcal{S} \subset \mathcal{A}$  be the subspace consisting of all  $\gamma$  whose image is contained in some open hemisphere (depending on  $\gamma$ ). Then the map  $\mathcal{S} \rightarrow \mathbf{S}^n$ ,  $\gamma \mapsto h_\gamma$ , defined in the preceding paragraph, is continuous.*

Before proving this, we record two results which we will use.

**(3.3) Lemma.** *Let  $C \subset \mathbf{S}^n$  be geodesically convex with non-empty interior. Then there exists a homeomorphism  $F: \mathbf{S}^{n-1} \rightarrow \partial C$  which is bi-Lipschitz.<sup>3</sup>*

*Proof.* We may assume without loss of generality that  $C$  contains  $N = e_{n+1}$  in its interior. Let

$$\{(p^1, \dots, p^{n+1}) \in \mathbf{S}^n : p^{n+1} = 0\}$$

be the equator of  $\mathbf{S}^n$ , which we identify with  $\mathbf{S}^{n-1}$ . Because  $N \in \text{Int}(C)$ , there exists  $\delta$ ,  $0 < \delta < 1$ , such that the open disk

$$U = \{(p^1, \dots, p^{n+1}) \in \mathbf{S}^n : (1 - \delta) < p^{n+1} \leq 1\} \quad (1)$$

is contained in  $C$ . In particular,  $\partial C \cap U = \emptyset$ . Since  $C$  cannot contain antipodal points,  $\partial C$  is also disjoint from  $-U$  (the image of  $U$  under the antipodal map). Because  $N \in C$  and  $-N \notin C$ , any semicircle containing them, say, the one that also contains  $\sigma \in \mathbf{S}^{n-1}$ , intersects  $\partial C$  at some point  $F(\sigma)$ .

Let  $p \in \partial C$ ,  $u \in U$ . We assert that the semicircle through  $p, u$  and  $-u$  cannot contain another point  $q \in \partial C$  (see fig. 4). If we take  $u = N$  then this shows that the definition of  $F: \mathbf{S}^{n-1} \rightarrow \partial C$  is unambiguous. Assume for a contradiction that the assertion is false, and suppose further that  $q$  lies between  $p$  and  $u$  (if it lies between  $-u$  and  $p$  instead, the argument is analogous). Consider the union of all geodesic segments joining points of  $U$  to  $p$ . This set contains  $q$  in its interior by hypothesis. The same is true of the union of all

<sup>3</sup>This means that there exist  $k_1, k_2 > 0$  such that

$$k_1 |\sigma - \tau| \leq |F(\sigma) - F(\tau)| \leq k_2 |\sigma - \tau| \text{ for any } \sigma, \tau \in \mathbf{S}^{n-1}.$$

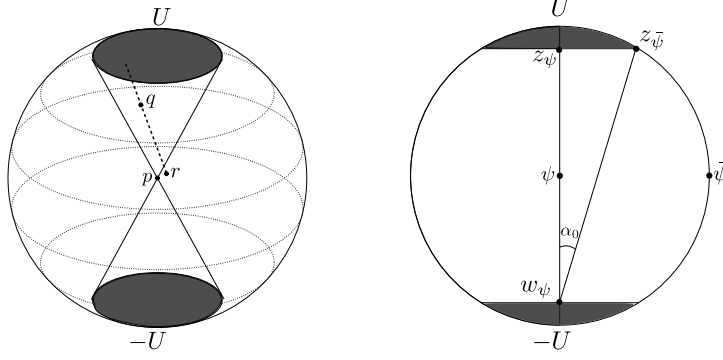


Figure 4: An illustration of part of the proof of (3.3) when  $n = 2$ .

geodesic segments joining points of  $U$  to  $r$ , whenever  $r$  is sufficiently close to  $p$ . Since  $p \in \partial C$ , we can choose  $r \in C$  to conclude from the convexity of  $C$  that  $q \in \text{Int}(C)$ , a contradiction.

Let  $\sigma \neq \tau \in \mathbf{S}^{n-1}$ . Then

$$\begin{aligned} \frac{|F(\sigma) - F(\tau)|}{|\sigma - \tau|} &\geq \frac{|(\sqrt{\delta(2-\delta)}\sigma, 1-\delta) - (\sqrt{\delta(2-\delta)}\tau, 1-\delta)|}{|\sigma - \tau|} \\ &\geq \sqrt{\delta(2-\delta)} > 0. \end{aligned}$$

Let  $d$  denote the distance function on  $\mathbf{S}^n$ . To establish a reverse Lipschitz condition for  $F$ , it suffices to prove that

$$\frac{d(F(\sigma), F(\tau))}{d(\sigma, \tau)} = \frac{F(\sigma)F(\tau)}{\angle F(\sigma)NF(\tau)}$$

admits an upper bound independent of the pair  $\sigma \neq \tau$ .<sup>4</sup> Since  $F(\sigma)F(\tau)$  is bounded by  $\pi$  and  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ , it actually suffices to establish a bound on

$$\frac{\sin(F(\sigma)F(\tau))}{\sin(\angle F(\sigma)NF(\tau))} = \frac{\sin(NF(\tau))}{\sin(\angle NF(\sigma)F(\tau))}, \quad (2)$$

where the equality follows from the law of sines (for spherical triangles) applied to  $\triangle F(\sigma)NF(\tau)$ . For arbitrary  $\psi \in \mathbf{S}^{n-1}$ , define  $z_\psi \in \partial U$  and  $w_\psi \in \partial(-U)$  to be the points where the great circle through  $N$  and  $\psi$  meets  $\partial U$  (resp.  $\partial(-U)$ ); more explicitly,

$$z_\psi = (\sqrt{\delta(2-\delta)}\psi, 1-\delta), \quad w_\psi = (\sqrt{\delta(2-\delta)}\psi, -1+\delta).$$

Let  $\psi, \bar{\psi} \in \mathbf{S}^{n-1}$  satisfy  $d(\psi, \bar{\psi}) = \frac{\pi}{2}$  and let  $\alpha_0$  be the angle at  $w_\psi$  in  $\triangle w_\psi z_\psi z_{\bar{\psi}}$ . Clearly, this angle is independent of  $\psi, \bar{\psi}$ . We claim that  $\angle NF(\sigma)F(\tau) > \alpha_0$ .

<sup>4</sup> $AB$  denotes the geodesic segment joining  $A$  to  $B$  and  $\angle ABC$  the angle at  $B$  in the spherical triangle  $ABC$ .

Otherwise, the geodesic through  $F(\sigma)$  and  $F(\tau)$  meets  $U$ , and so does the geodesic through  $p$  and  $F(\tau)$  for  $p$  close to  $F(\sigma)$ , for  $U$  is open. Since  $F(\sigma) \in \partial C$ , we can choose  $p \in C$  with this property, which, using the convexity of  $C$ , contradicts the fact that  $F(\tau) \notin \text{Int}(C)$ . Hence, we can complete (2) to

$$\frac{\sin(F(\sigma)F(\tau))}{\sin(\angle F(\sigma)NF(\tau))} = \frac{\sin(NF(\tau))}{\sin(\angle NF(\sigma)F(\tau))} < \frac{\pi}{\sin \alpha_0},$$

finishing the proof that  $F$  is bi-Lipschitz.  $\square$

**(3.4) Lemma.** *Let  $A \subset \mathbf{S}^n$  be a closed set of Hausdorff dimension less than  $n$ . If  $B_\varepsilon$  consists of all points at distance less than  $\varepsilon$  from  $A$ , then  $\lim_{\varepsilon \rightarrow 0} V(B_\varepsilon) = 0$ , where  $V$  denotes the volume in  $\mathbf{S}^n$ .*

*Proof.* Let  $\delta(S)$  denote the diameter of a set  $S \subset \mathbf{S}^n$  and  $\Gamma_\alpha(S)$  its Hausdorff measure of dimension  $\alpha > 0$ . Since  $\Gamma_n(A) = 0$ , given any  $\eta > 0$  we can cover  $A$  by a countable collection of sets  $A_k \subset \mathbf{S}^n$  such that  $\sum_k \delta(A_k)^n < \eta$ . Each  $A_k$  can be enclosed in an open ball  $U_k$  of diameter  $3\delta(A_k)$ , and since  $A$  is compact,  $\bigcup_k U_k$  contains some  $B_\varepsilon$ . Therefore, the conclusion follows from the estimate

$$V(B_\varepsilon) \leq \sum_k V(U_k) \leq C \sum_k \delta(U_k)^n < 3^n C \eta,$$

where  $C$  is the constant, depending only on  $n$ , which relates the Hausdorff measure in dimension  $n$  to the usual measure (volume).  $\square$

*Proof of (3.2).* It suffices to prove the result when  $\mathcal{A}$  has the  $C^0$  topology, since it is coarser than the  $C^r$  topology for any  $r \geq 1$ .

Let  $\gamma \in \mathcal{S}$  and  $\mathcal{H}$  (regarded as a subset of  $\mathbf{S}^n$ ) be the set of all open hemispheres containing  $\gamma([0, 1])$ . Let  $\varepsilon > 0$  and define

$$B_\varepsilon = \bigcup_{q \in \partial \mathcal{H}} B(q; \varepsilon), \quad \mathcal{H}_0 = \mathcal{H} \setminus B_\varepsilon \quad \text{and} \quad \mathcal{H}_1 = \mathcal{H} \cup B_\varepsilon. \quad (3)$$

Then  $\overline{\mathcal{H}}_0 \subset \mathcal{H} \subset \overline{\mathcal{H}} \subset \mathcal{H}_1$ . As a consequence of the compactness of  $[0, 1]$ ,  $\overline{\mathcal{H}}_0$  and  $\mathbf{S}^n \setminus \mathcal{H}_1$ , there exists  $\delta > 0$  for which

$$\langle \gamma(t), u \rangle \geq \delta \text{ if } u \in \mathcal{H}_0 \quad \text{and} \quad \langle \gamma(t), v \rangle \leq -\delta \text{ for } v \notin \mathcal{H}_1 \quad \text{for all } t \in [0, 1].$$

Consequently, there exists a neighborhood  $\mathcal{U} \subset \mathcal{A}$  of  $\gamma$  such that if  $\eta \in \mathcal{U}$  then

$$\langle \eta(t), u \rangle \geq \delta/2 \text{ if } u \in \mathcal{H}_0 \quad \text{and} \quad \langle \eta(t), v \rangle \leq -\delta/2 \text{ for } v \notin \mathcal{H}_1 \quad \text{for all } t \in [0, 1].$$

Thus, if  $\mathcal{K}$  is the set of hemispheres containing  $\eta$ , we have  $\overline{\mathcal{H}}_0 \subset \mathcal{K} \subset \overline{\mathcal{K}} \subset \mathcal{H}_1$ .



Without loss of generality, we may assume that the barycenter  $h_\gamma$  of  $\mathcal{H}$  is  $e_{n+1}$ . Let  $h_\eta^j$  denote the  $j$ -th coordinate of the barycenter  $h_\eta$  of  $\mathcal{K}$ . By definition  $h_\eta^j \int_{\mathcal{K}} dx = \int_{\mathcal{K}} x_j dx$ , and the latter term satisfies

$$\begin{aligned} \int_{\mathcal{K}} x_j dx &= \int_{\mathcal{H}} x_j dx + \int_{\mathcal{K} \setminus \mathcal{H}} x_j dx - \int_{\mathcal{H} \setminus \mathcal{K}} x_j dx \\ &\leq \int_{\mathcal{H}} x_j dx + \int_{\mathcal{H}_1 \setminus \mathcal{H}_0} 1 dx - \int_{\mathcal{H}_1 \setminus \mathcal{H}_0} (-1) dx \\ &= \int_{\mathcal{H}} x_j dx + 2 \int_{\mathcal{H}_1 \setminus \mathcal{H}_0} dx \end{aligned}$$

Since the  $j$ -th coordinate  $h_\gamma^j$  of  $h_\gamma$  is non-negative for each  $j$ , it follows that

$$h_\eta^j \leq \left( \frac{\int_{\mathcal{H}} dx}{\int_{\mathcal{H}_0} dx} \right) h_\gamma^j + 2 \left( \frac{\int_{\mathcal{H}_1 \setminus \mathcal{H}_0} dx}{\int_{\mathcal{H}_0} dx} \right);$$

similarly,

$$h_\eta^j \geq \left( \frac{\int_{\mathcal{H}} dx}{\int_{\mathcal{H}_1} dx} \right) h_\gamma^j - 2 \left( \frac{\int_{\mathcal{H}_1 \setminus \mathcal{H}_0} dx}{\int_{\mathcal{H}_0} dx} \right).$$

The set  $\partial\mathcal{H}$  has Hausdorff dimension  $n - 1$ , for it is the image of  $\mathbf{S}^{n-1}$  under a Lipschitz map (by (3.1) and (3.3)). We also have:

$$\int_{\mathcal{H}_1} dx \leq \int_{\mathcal{H}} dx + V(B_\varepsilon), \quad \int_{\mathcal{H}_0} dx \geq \int_{\mathcal{H}} dx - V(B_\varepsilon) \quad \text{and} \quad \int_{\mathcal{H}_1 \setminus \mathcal{H}_0} dx \leq V(B_\varepsilon).$$

Therefore, according to (3.4), we can make  $h_\eta$  arbitrarily close to  $h_\gamma$  for all  $\eta \in \mathcal{U}$  by an adequate choice of  $\varepsilon$  in (3). In other words,  $\gamma \mapsto h_\gamma$  is continuous.  $\square$

The following result (for  $C^1$  curves) is quite old; see [5], §1.

**(3.5) Lemma.** *Let  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  be an admissible closed curve, and let  $\mathbf{t}(t)$  denote its unit tangent vector at  $\gamma(t)$ . Then the curve  $\mathbf{t}: [0, 1] \rightarrow \mathbf{S}^2$  intersects any great circle.*

*Proof.* Let  $L$  be the length of  $\gamma$  and  $h \in \mathbf{S}^2$  any fixed vector. Since  $\gamma$  is a closed curve,

$$\int_0^L \langle \mathbf{t}(s), h \rangle ds = \int_0^L \langle \gamma'(s), h \rangle ds = \langle \gamma(L) - \gamma(0), h \rangle = 0.$$

In particular, the function  $\langle \mathbf{t}(s), h \rangle$  must vanish for some  $s_0 \in [0, L]$ . This means that  $\mathbf{t}$  intersects the great circle  $C = \{p \in \mathbf{S}^2 : \langle p, h \rangle = 0\}$  at  $\mathbf{t}(s_0)$ .  $\square$

## 4

### The Connected Components of $\mathcal{L}_{\kappa_1}^{\kappa_2}$

The following theorem is the main result of this work. It presents a description of the components of  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  in terms of  $\kappa_1$  and  $\kappa_2$ .

**(4.1) Theorem.** *Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ ,  $\rho_i = \operatorname{arccot} \kappa_i$  ( $i = 1, 2$ ) and  $\lfloor x \rfloor$  denote the greatest integer smaller than or equal to  $x$ . Then  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  has exactly  $n$  connected components  $\mathcal{L}_1, \dots, \mathcal{L}_n$ , where*

$$n = \left\lfloor \frac{\pi}{\rho_1 - \rho_2} \right\rfloor + 1 \quad (1)$$

and  $\mathcal{L}_j$  contains circles traversed  $j$  times ( $1 \leq j \leq n$ ). The component  $\mathcal{L}_{n-1}$  also contains circles traversed  $(n-1) + 2k$  times, and  $\mathcal{L}_n$  contains circles traversed  $n + 2k$  times, for  $k \in \mathbf{N}$ . Moreover, each of  $\mathcal{L}_1, \dots, \mathcal{L}_{n-2}$  is homotopy equivalent to  $\mathbf{SO}_3$  ( $n \geq 3$ ).

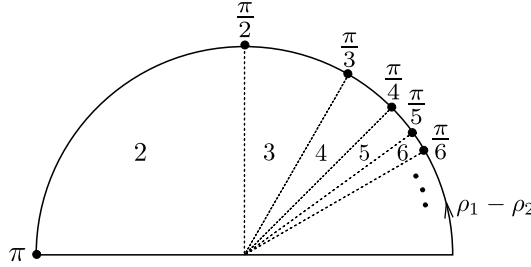


Figure 5: The number of connected components of  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ , as  $\rho_1 - \rho_2$  varies in  $(0, \pi]$  (where  $\rho_i = \operatorname{arccot} \kappa_i$ ). When  $\rho_1 - \rho_2 = \frac{\pi}{n}$ ,  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  has  $n + 1$  components.

If we replace  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  by  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  in the statement then the conclusion is the same, except that  $\mathcal{L}_1(I), \dots, \mathcal{L}_{n-2}(I)$  are now contractible, and, of course, the circles are required to have initial and final frames equal to  $I$ . This is what will actually be proved; the theorem follows from this and the homeomorphism  $\mathcal{L}_{\kappa_1}^{\kappa_2} \approx \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ , which was established in (2.15). We could also have replaced  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  by the space of all  $C^r$  closed curves ( $r \geq 2$ ) whose geodesic curvatures lie in the interval  $(\kappa_1, \kappa_2)$ , with the  $C^r$  topology, since this space is homotopy equivalent to the former, by (2.10).

*Examples.* Let us first discuss some concrete cases of the theorem.

(a) We have already mentioned (on p. 22) that  $\mathcal{L}_{-\infty}^{+\infty} = \mathcal{I} \simeq \mathbf{SO}_3 \times (\Omega\mathbf{S}^3 \sqcup \Omega\mathbf{S}^3)$  has two connected components  $\mathcal{I}_+$  and  $\mathcal{I}_-$ , which are characterized by:  $\gamma \in \mathcal{I}_+$  if and only if  $\tilde{\Phi}_\gamma(1) = \tilde{\Phi}_\gamma(0)$  and  $\gamma \in \mathcal{I}_-$  if and only if  $\tilde{\Phi}_\gamma(1) = -\tilde{\Phi}_\gamma(0)$ . This is consistent with (4.1).

(b) Suppose  $\kappa_0 < 0$ . Setting  $\rho_2 = 0$  and  $\rho_1 = \operatorname{arccot} \kappa_0$  in (4.1), we find that  $\mathcal{L}_{\kappa_0}^{+\infty}$  also has two connected components. Since  $\mathcal{L}_{\kappa_0}^{+\infty}$  can be considered a subspace of  $\mathcal{L}_{-\infty}^{+\infty}$ , these components have the same characterization in terms of  $\tilde{\Phi}(1)$ : two curves  $\gamma, \eta \in \mathcal{L}_{\kappa_0}^{+\infty}$  are homotopic if and only if  $\tilde{\Phi}_\gamma(1) = \pm \tilde{\Phi}_\eta(0)$  and  $\tilde{\Phi}_\eta(1) = \pm \tilde{\Phi}_\gamma(0)$ , with the same choice of sign for both curves.

(c) In contrast,  $\mathcal{L}_{\kappa_0}^{+\infty}$  has at least three connected components when  $\kappa_0 \geq 0$ . It has exactly three components in case

$$0 \leq \kappa_0 < \frac{1}{\sqrt{3}}.$$

The case  $\kappa_0 = 0$  is Little's theorem ([8], thm. 1). If

$$\frac{1}{\sqrt{3}} \leq \kappa_0 < 1$$

it has four connected components and so forth.

To sum up, as we impose starker restrictions on the geodesic curvatures, a homotopy which existed “before” may now be impossible to carry out. For instance, in any space  $\mathcal{L}_{\kappa_0}^{+\infty}$  with  $\kappa_0 < 0$ , it is possible to deform a circle traversed once into a circle traversed three times. However, in  $\mathcal{L}_0^{+\infty}$  this is not possible anymore, which gives rise to a new component.

The first part of theorem (4.1) is an immediate consequence of the following results.

**(4.2) Theorem.** *Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ . Every curve in  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  (resp.  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ ) lies in the same component as a circle traversed  $k$  times, for some  $k \in \mathbf{N}$  (depending on the curve).*

**(4.3) Theorem.** *Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$  and let  $\sigma_k \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  (resp.  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ ) denote any circle traversed  $k \geq 1$  times. Then  $\sigma_k, \sigma_{k+2}$  lie in the same component of  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  (resp.  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ ) if and only if*

$$k \geq \left\lceil \frac{\pi}{\rho_1 - \rho_2} \right\rceil \quad (\rho_i = \operatorname{arccot} \kappa_i, \ i = 1, 2).$$

The following very simple result will be used implicitly in the sequel; it implies in particular that it does not matter which circle  $\sigma_k$  we choose in (4.2) and (4.3).

**(4.4) Lemma.** *Let  $\sigma, \tilde{\sigma} \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  (resp.  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ ) be parametrized circles traversed the same number of times. Then  $\sigma$  and  $\tilde{\sigma}$  lie in the same connected component of  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  (resp.  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ ).*

*Proof.* By (2.15), it suffices to prove the result for  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ , since any circle in  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  is obtained from a circle in the former space by a rotation and  $\mathbf{SO}_3$  is connected. By (2.1), we can assume that both  $\sigma$  and  $\tilde{\sigma}$  are parametrized by a multiple of arc-length. Let  $k$  be the common number of times that the circles are traversed, let  $\rho, \tilde{\rho} \in (\rho_2, \rho_1)$  be their respective radii of curvature (where  $\rho_i = \operatorname{arccot}(\kappa_i)$ ) and define  $\rho(s) = (1-s)\rho + s\tilde{\rho}$  for  $s \in [0, 1]$ . Then

$$(s, t) \mapsto \cos \rho(s) (\cos \rho(s), 0, \sin \rho(s)) \\ + \sin \rho(s) (\sin \rho(s) \cos(2k\pi t), \sin(2k\pi t), -\cos \rho(s) \cos(2k\pi t)),$$

where  $s, t \in [0, 1]$ , yields the desired homotopy between  $\sigma$  and  $\tilde{\sigma}$  in  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ .  $\square$

Next we introduce the main concepts and tools used in the proofs of the theorems listed above. From now on we shall work almost exclusively with spaces of type  $\mathcal{L}_{\kappa_0}^{+\infty}$  and  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ ; we are allowed to do so by (2.25).

### The bands spanned by a curve

Let  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  be a  $C^2$  regular curve. For  $t \in [0, 1]$ , let  $\chi(t)$  (or  $\chi_\gamma(t)$ ) be the center, on  $\mathbf{S}^2$ , of the osculating circle to  $\gamma$  at  $\gamma(t)$ .<sup>1</sup> The point  $\chi(t)$  will be called the *center of curvature* of  $\gamma$  at  $\gamma(t)$ , and the correspondence  $t \mapsto \chi(t)$  defines a new curve  $\chi: [0, 1] \rightarrow \mathbf{S}^2$ , the *caustic* of  $\gamma$ . In symbols,

$$\chi(t) = \cos \rho(t) \gamma(t) + \sin \rho(t) \mathbf{n}(t). \quad (2)$$

Here, as always,  $\rho = \operatorname{arccot} \kappa$  is the radius of curvature and  $\mathbf{n}$  the unit normal to  $\gamma$ . Note that the caustic of a circle degenerates to a single point, its center. This is explained by the following result.

**(4.5) Lemma.** *Let  $r \geq 2$ ,  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  be a  $C^r$  regular curve and  $\chi$  its caustic. Then  $\chi$  is a curve of class  $C^{r-2}$ . When  $\chi$  is differentiable,  $\dot{\chi}(t) = 0$  if and only if  $\dot{\kappa}(t) = 0$ , where  $\kappa$  is the geodesic curvature of  $\gamma$ .*

*Proof.* If  $\gamma$  is  $C^r$  then  $\rho$  is a  $C^{r-2}$  function, hence  $\chi$  is also of class  $C^{r-2}$ . The proof of the second assertion is a straightforward computation: Using the

<sup>1</sup>There are two possibilities for the center on  $\mathbf{S}^2$  of a circle. To distinguish them we use the orientation of the circle, as in fig. 2. The radius of curvature  $\rho(t)$  is the distance from  $\gamma(t)$  to the center  $\chi(t)$ , measured along  $\mathbf{S}^2$ .

arc-length parameter  $s$  of  $\gamma$  instead of  $t$ , we find that

$$\begin{aligned}\chi'(s) &= \rho'(s) \left( -\sin \rho(s) \gamma(s) + \cos \rho(s) \mathbf{n}(s) \right) + \left( \cos \rho(s) - \kappa(s) \sin \rho(s) \right) \mathbf{t}(s) \\ &= \frac{\kappa'(s)}{1 + \kappa(s)^2} \left( \sin \rho(s) \gamma(s) - \cos \rho(s) \mathbf{n}(s) \right),\end{aligned}$$

where we have used that

$$\cos \rho - \kappa \sin \rho = \sin \rho (\cot \rho - \kappa) = 0$$

together with  $0 < \rho < \pi$ . Therefore,  $\chi'(s) = 0$  if and only if  $\kappa'(s)$  vanishes.  $\square$

**(4.6) Definitions.** Let  $\kappa_0 \in \mathbf{R}$ ,  $\rho_0 = \operatorname{arccot} \kappa_0$  and  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ . Define the *regular band*  $B_\gamma$  and the *caustic band*  $C_\gamma$  to be the maps

$$B_\gamma: [0, 1] \times [\rho_0 - \pi, 0] \rightarrow \mathbf{S}^2 \quad \text{and} \quad C_\gamma: [0, 1] \times [0, \rho_0] \rightarrow \mathbf{S}^2$$

given by the same formula:

$$(t, \theta) \mapsto \cos \theta \gamma(t) + \sin \theta \mathbf{n}(t). \quad (3)$$

The image of  $C_\gamma$  will be denoted by  $C$ , and the geodesic circle orthogonal to  $\gamma$  at  $\gamma(t)$  will be denoted by  $\Gamma_t$ . As a set,

$$\Gamma_t = \{ \cos \theta \gamma(t) + \sin \theta \mathbf{n}(t) : \theta \in [-\pi, \pi) \}.$$

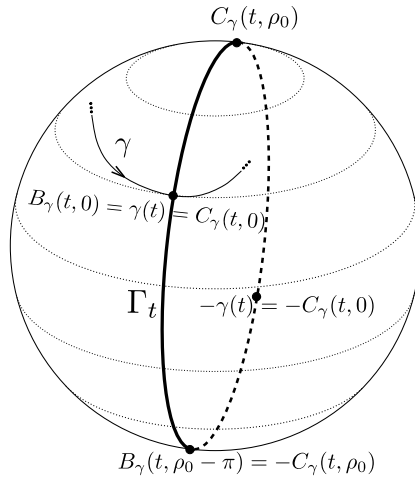


Figure 6:

For fixed  $t$ , the images of  $\pm B_\gamma(t, \cdot)$  and  $\pm C_\gamma(t, \cdot)$  divide the circle  $\Gamma_t$  in four parts. Note also that  $\chi_\gamma(t) = C_\gamma(t, \rho(t))$ .

**(4.7) Lemma.** *Let  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  and let  $B_\gamma: [0, 1] \times [\rho_0 - \pi, 0] \rightarrow \mathbf{S}^2$  be the regular band spanned by  $\gamma$ . Then:*

- (a) *The derivative of  $B_\gamma$  is an isomorphism at every point.*
- (b)  *$\frac{\partial B_\gamma}{\partial \theta}(t, \theta)$  has norm 1 and is orthogonal to  $\frac{\partial B_\gamma}{\partial t}(t, \theta)$ . Moreover,*

$$\det \left( B_\gamma, \frac{\partial B_\gamma}{\partial t}, \frac{\partial B_\gamma}{\partial \theta} \right) > 0.$$

- (c)  *$C_\gamma$  fails to be an immersion precisely at the points  $(t, \rho(t))$  whose images form the caustic  $\chi$ .*

*Proof.* We have:

$$\frac{\partial B_\gamma}{\partial \theta}(t, \theta) = -\sin \theta \gamma(t) + \cos \theta \mathbf{n}(t). \quad (4)$$

and

$$\frac{\partial B_\gamma}{\partial t}(t, \theta) = |\dot{\gamma}(t)| (\cos \theta - \kappa(t) \sin \theta) \mathbf{t}(t) \quad (5)$$

$$= \frac{|\dot{\gamma}(t)|}{\sin \rho(t)} \sin(\rho(t) - \theta) \mathbf{t}(t), \quad (6)$$

where  $\rho(t) = \operatorname{arccot} \kappa(t)$  is the radius of curvature of  $\gamma$  at  $\gamma(t)$ . The inequality  $\kappa_0 < \kappa < +\infty$  translates into  $0 < \rho < \rho_0$ , hence the factor multiplying  $\mathbf{t}(t)$  in (6) is positive for  $\theta$  satisfying  $\rho_0 - \pi \leq \theta \leq 0$ , and this implies (a) and (b). Part (c) also follows directly from (6), because  $C_\gamma$  and  $B_\gamma$  are defined by the same formula.  $\square$

Thus,  $B_\gamma$  is an immersion (and a submersion) at every point of its domain. It is merely a way of collecting the regular translations of  $\gamma$  (as defined on p. 24) in a single map.

If we fix  $t$  and let  $\theta$  vary in  $(0, \rho_0)$ , the section  $C_\gamma(t, \theta)$  of  $\Gamma_t$  describes the set of “valid” centers of curvature for  $\gamma$  at  $\gamma(t)$ , in the sense that the circle centered at  $C_\gamma(t, \theta)$  passing through  $\gamma(t)$ , with the same orientation, has geodesic curvature greater than  $\kappa_0$ . This interpretation is important because it motivates many of the constructions that we consider ahead.

### Condensed and diffuse curves

**(4.8) Definition.** Let  $\kappa_0 \in \mathbf{R}$  and  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ . We shall say that  $\gamma$  is *condensed* if the image  $C$  of  $C_\gamma$  is contained in a closed hemisphere, and *diffuse* if  $C$  contains antipodal points (i.e., if  $C \cap -C \neq \emptyset$ ).

*Examples.* A circle in  $\mathcal{L}_{\kappa_0}^{+\infty}$  is always condensed for  $\kappa_0 \geq 0$ , but when  $\kappa_0 < 0$  it may or may not be condensed, depending on its radius. If a curve contains antipodal points then it must be diffuse, since  $C_\gamma(t, 0) = \gamma(t)$ . By the same reason, a condensed curve is itself contained in a closed hemisphere.

There exist curves which are condensed and diffuse at the same time; an example is a geodesic circle in  $\mathcal{L}_{\kappa_0}^{+\infty}$ , with  $\kappa_0 < 0$ . There also exist curves which are neither condensed nor diffuse. To see this, let  $\mathbf{S}^1$  be identified with the equator of  $\mathbf{S}^2$  and let  $\zeta \in \mathbf{S}^1$  be a primitive third root of unity. Choose small neighborhoods  $U_i$  of  $\zeta^i$  ( $i = 0, 1, 2$ ) and  $V$  of the north pole in  $\mathbf{S}^2$ . Then the set  $G$  consisting of all geodesic segments joining points of  $U_1 \cup U_2 \cup U_3$  to points of  $V$  does not contain antipodal points, nor is it contained in a closed hemisphere, by (11.2). By taking  $\rho_0 = \operatorname{arccot} \kappa_0$  to be very small, we can construct a curve  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  for which  $C = \operatorname{Im}(C_\gamma) \subset G$ , but  $\zeta^i \in C$  for each  $i$ , so that  $\gamma$  is neither condensed nor diffuse.

To sum up, a curve may be condensed, diffuse, neither of the two, or both simultaneously, but this ambiguity is not as important as it seems.

**(4.9) Lemma.** *Let  $\kappa_0 \in \mathbf{R}$  and suppose that  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  is condensed. Then the image of  $\chi = \chi_\gamma$  is contained in an open hemisphere.*

*Proof.* Let  $H = \{p \in \mathbf{S}^2 : \langle p, h \rangle \geq 0\}$  be a closed hemisphere containing the image of  $C_\gamma$  and suppose that  $\langle \chi(t_0), h \rangle = 0$  for some  $t_0 \in [0, 1]$ . At least one of  $\gamma(t_0)$  or  $\mathbf{n}(t_0)$  is not a multiple of  $h \times \chi_\gamma(t_0)$ . In either case,

$$C_\gamma((t_0 - \varepsilon, t_0 + \varepsilon) \times (\rho(t_0) - \varepsilon, \rho(t_0) + \varepsilon)) \not\subset H,$$

for sufficiently small  $\varepsilon > 0$ , a contradiction.  $\square$

Let  $\kappa_0 \in \mathbf{R}$  and let  $\mathcal{O} \subset \mathcal{L}_{\kappa_0}^{+\infty}$  denote the subset of condensed curves. Define a map  $h: \mathcal{O} \rightarrow \mathbf{S}^2$  by  $\gamma \mapsto h_\gamma$ , where  $h_\gamma$  is the image under gnomonic (central) projection of the barycenter, in  $\mathbf{R}^3$ , of the set of closed hemispheres which contain  $C = \operatorname{Im}(C_\gamma)$ .

**(4.10) Lemma.** *The map  $h: \mathcal{O} \rightarrow \mathbf{S}^2$ ,  $\gamma \mapsto h_\gamma$ , defined above is continuous.*

*Proof.* Consider first the subset  $\mathcal{S} \subset \mathcal{L}_{\kappa_0}^{+\infty}$  consisting of all curves  $\gamma$  such that  $\operatorname{Im}(C_\gamma)$  is contained in an open hemisphere. A minor modification in the proof of (3.1) shows that, in this case, the set  $\mathcal{H}$  of closed hemispheres which contain  $\gamma$  is geodesically convex, open and contained in an open hemisphere. Thus, we may apply (3.3) and (3.4) to  $\mathcal{H}$  and  $\partial\mathcal{H}$ , respectively. Using these, the proof of (3.2) goes through almost unchanged to establish that the restriction of  $h$  to  $\mathcal{S}$  is continuous.

It remains to prove that  $h$  is continuous at any curve  $\gamma \in \mathcal{O} \setminus \mathcal{S}$ . Note first that there exists exactly one closed hemisphere  $h_\gamma$  containing  $\text{Im}(C_\gamma)$  in this case. For if  $C = \text{Im}(C_\gamma)$  is contained in distinct closed hemispheres  $H_1$  and  $H_2$ , then it is contained in the closed lune  $H_1 \cap H_2$ . The boundary of  $\text{Im}(C_\gamma)$  is contained in the union of the images of  $\gamma = C_\gamma(\cdot, 0)$  and  $\tilde{\gamma} = C_\gamma(\cdot, \rho_0)$ ; since these curves have a unit tangent vector at all points, they cannot pass through either of the points in  $E_1 \cap E_2$  (where  $E_i$  is the equator corresponding to  $H_i$ ). It follows that  $\text{Im}(C_\gamma)$  is contained in an open hemisphere, a contradiction. Furthermore, by (11.1), (11.2) and (11.5), we can find

$$z_i = C_\gamma(t_i, \theta_i) \in \text{Im}(C_\gamma) \cap \{p \in \mathbf{S}^2 : \langle p, h_\gamma \rangle = 0\} \quad (\theta_i \in \{0, \rho_0\}, \quad i = 1, 2, 3)$$

such that 0 lies in the simplex spanned by  $z_1, z_2, z_3$ ; any hemisphere other than  $\pm h_\gamma$  separates these three points. Let  $z_0 = C_\gamma(t_0, \theta_0)$  be a point in  $\text{Im}(C_\gamma)$  satisfying  $\langle z_0, h_\gamma \rangle > 0$ . Then we may choose  $\delta > 0$  and a sufficiently small neighborhood  $\mathcal{U}$  of  $\gamma$  in  $\mathcal{L}_{\kappa_0}^{+\infty}$  such that  $\langle C_\eta(t_0, \theta_0), k \rangle < 0$  for any  $\eta \in \mathcal{U}$  and  $k \in \mathbf{S}^2$  satisfying  $d(k, h_\gamma) \geq \pi - \delta$  (where  $d$  denotes the distance function on  $\mathbf{S}^2$ ). By reducing  $\mathcal{U}$  if necessary, we can also arrange that if  $\delta \leq d(k, h_\gamma) \leq \pi - \delta$ , then the hemisphere corresponding to  $k$  separates  $\{C_\eta(t_i, \theta_i), \quad i = 1, 2, 3\}$  whenever  $\eta \in \mathcal{U}$ . The conclusion is that if  $k \in \mathbf{S}^2$  satisfies  $\langle c, k \rangle \geq 0$  for all  $c \in \text{Im}(C_\eta)$  and  $\eta \in \mathcal{U}$ , then  $d(k, h_\gamma) < \delta$ . It follows that  $h$  is continuous at  $\gamma \in \mathcal{O} \setminus \mathcal{S}$ .  $\square$

An argument entirely similar to that given above can be used to modify (3.2) as follows.

**(4.11) Lemma.** *Let  $\kappa_0 \in \mathbf{R}$  and  $\mathcal{H} \subset \mathcal{L}_{\kappa_0}^{+\infty}$  be the subspace consisting of all  $\gamma$  whose image is contained in some closed hemisphere (depending on  $\gamma$ ). Then the map  $h: \mathcal{H} \rightarrow \mathbf{S}^2$ , which associates to  $\gamma$  the barycenter  $h_\gamma$  on  $\mathbf{S}^2$  of the set of closed hemispheres that contain  $\gamma$ , is continuous.*  $\square$



## 5 Grafting

**(5.1) Definition.** Let  $\gamma: [a, b] \rightarrow \mathbf{S}^2$  be an admissible curve. The *total curvature*  $\text{tot}(\gamma)$  of  $\gamma$  is given by

$$\text{tot}(\gamma) = \int_a^b K(t) |\dot{\gamma}(t)| dt,$$

where

$$K = \sqrt{1 + \kappa^2} = \csc \rho \quad (1)$$

is the Euclidean curvature of  $\gamma$ . We say that  $\gamma: [0, T] \rightarrow \mathbf{S}^2$ ,  $u \mapsto \gamma(u)$ , is a *parametrization of  $\gamma$  by curvature* if

$$|\Phi'_\gamma(u)| = \sqrt{2} \text{ or, equivalently, } |\tilde{\Phi}'_\gamma(u)| = \frac{1}{2} \text{ for a.e. } u \in [0, T].$$

The equivalence of the two equalities comes from (2.11). The next result justifies our terminology.

**(5.2) Lemma.** *Let  $\gamma: [0, T] \rightarrow \mathbf{S}^2$  be an admissible curve. Then:*

(a)  *$\gamma$  is parametrized by curvature if and only if*

$$\text{tot}(\gamma|_{[0, u]}) = u \text{ for every } u \in [0, T].$$

(b) *If  $\gamma$  is parametrized by curvature then its logarithmic derivatives  $\Lambda = \Phi_\gamma^{-1}\Phi'_\gamma$  and  $\tilde{\Lambda} = \tilde{\Phi}_\gamma^{-1}\tilde{\Phi}'$  are given by:*

$$\Lambda(u) = \begin{pmatrix} 0 & -\sin \rho(u) & 0 \\ \sin \rho(u) & 0 & -\cos \rho(u) \\ 0 & \cos \rho(u) & 0 \end{pmatrix},$$

$$\tilde{\Lambda}(u) = \frac{1}{2}(\cos \rho(u)\mathbf{i} + \sin \rho(u)\mathbf{k}).$$

Here, as always,  $\rho$  is the radius of curvature of  $\gamma$ . In the expression for  $\tilde{\Lambda}$  above and in the sequel we are identifying the Lie algebra  $\tilde{\mathfrak{so}}_3 = T_1\mathbf{S}^3$  (the tangent space to  $\mathbf{S}^3$  at  $\mathbf{1}$ ) with the vector space of all imaginary quaternions.

Also, it follows from (a) that if  $\gamma: [0, T] \rightarrow \mathbf{S}^2$  is parametrized by curvature then  $T = \text{tot}(\gamma)$ .

*Proof.* Let us denote differentiation with respect to  $u$  by  $'$ . Using (1), we deduce that

$$\Lambda(u) = |\gamma'(u)| \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\kappa(u) \\ 0 & \kappa(u) & 0 \end{pmatrix} \quad (2)$$

$$= K(u) |\gamma'(u)| \begin{pmatrix} 0 & -\sin \rho(u) & 0 \\ \sin \rho(u) & 0 & -\cos \rho(u) \\ 0 & \cos \rho(u) & 0 \end{pmatrix}, \quad (3)$$

hence  $|\Phi'(u)| = |\Lambda(u)| = \sqrt{2} K(u) |\gamma'(u)|$ . Therefore,  $\gamma$  is parametrized by curvature if and only if

$$K(u) |\gamma'(u)| = 1 \text{ for a.e. } u \in [0, T].$$

Integrating we deduce that this is equivalent to

$$\text{tot}(\gamma|_{[0, u]}) = u \text{ for every } u \in [0, T],$$

which proves (a). The expression for  $\tilde{\Lambda}$  is obtained from (2), using that under the isomorphism  $\tilde{\mathfrak{so}}_3 \rightarrow \mathfrak{so}_3$  induced by the projection  $\mathbf{S}^3 \rightarrow \mathbf{SO}_3$ ,  $\frac{i}{2}$ ,  $\frac{j}{2}$  and  $\frac{k}{2}$  correspond respectively to

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \square$$

We now introduce the essential notion of grafting.

**(5.3) Definition.** Let  $\gamma_i: [0, T_i] \rightarrow \mathbf{S}^2$  ( $i = 0, 1$ ) be admissible curves parametrized by curvature.

(a) A *grafting function* is a function  $\phi: [0, s_0] \rightarrow [0, s_1]$  of the form

$$\phi(t) = t + \sum_{x < t, x \in X^+} \delta^+(x) + \sum_{x \leq t, x \in X^-} \delta^-(x), \quad (4)$$

where  $X^+ \subset [0, s_0]$  and  $X^- \subset [0, s_0]$  are countable sets and  $\delta^\pm: X^\pm \rightarrow (0, +\infty)$  are arbitrary functions.

(b) We say that  $\gamma_1$  is *obtained from  $\gamma_0$  by grafting*, denoted  $\gamma_0 \preceq \gamma_1$ , if there exists a grafting function  $\phi: [0, T_0] \rightarrow [0, T_1]$  such that  $\Lambda_{\gamma_0} = \Lambda_{\gamma_1} \circ \phi$ .

(c) Let  $J$  be an interval (not necessarily closed). A *chain of grafts* consists of a homotopy  $s \mapsto \gamma_s$ ,  $s \in J$ , and a family of grafting functions  $\phi_{s_0, s_1}: [0, s_0] \rightarrow [0, s_1]$ ,  $s_0 < s_1 \in J$ , such that:

- (i)  $\Lambda_{\gamma_{s_0}} = \Lambda_{\gamma_{s_1}} \circ \phi_{s_0, s_1}$  whenever  $s_0 < s_1$ ;
- (ii)  $\phi_{s_0, s_2} = \phi_{s_1, s_2} \circ \phi_{s_0, s_1}$  whenever  $s_0 < s_1 < s_2$ .

Here every curve is admissible and parametrized by curvature.

**(5.4) Remarks.**

(a) A function  $\phi: [0, s_0] \rightarrow [0, s_1]$ ,  $s_0 \leq s_1$ , is a grafting function if and only if it is increasing and there exists a countable set  $X \subset [0, s_0]$  such that  $\phi(t) = t + c$  whenever  $t$  belongs to one of the intervals which form  $(0, s_0) \setminus X$ , where  $c \geq 0$  is a constant depending on the interval.

(b) Observe that in eq. (4),  $x < t$  in the first sum, while  $x \leq t$  in the second sum. We do not require  $X^+$  and  $X^-$  to be disjoint, and they may be finite (or even empty).

(c) If  $\phi: [0, s_0] \rightarrow [0, s_1]$  is a grafting function then it is monotone increasing and has derivative equal to 1 a.e.. Moreover,  $\phi(t+h) - \phi(t) \geq h$  for any  $t$  and  $h \geq 0$ ; in particular,  $s_0 \leq s_1$ .

(d) As the name suggests,  $\gamma_0 \preceq \gamma_1$  if  $\gamma_1$  is obtained by inserting a countable number of pieces of curves (e.g., arcs of circles) at chosen points of  $\gamma_0$  (see fig. 9). This can be used, for instance, to increase the total curvature of a curve. The difficulty is that it is usually not clear how we can graft pieces of curves onto a closed curve so that the resulting curve is still closed and the restrictions on the geodesic curvature are not violated.

(e) Two curves  $\gamma_0, \gamma_1 \in \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$  agree if and only if  $\Lambda_{\gamma_0} = \Lambda_{\gamma_1}$  a.e. on  $[0, 1]$ . Indeed,  $\gamma_i = \Phi_{\gamma_i} e_1$ , where  $\Phi_{\gamma_i}$  is the unique solution to an initial value problem as in eq. (4) of §1. Of course, if the curves are parametrized by curvature instead, then the latter condition should be replaced by  $T_0 = T_1$  and  $\Lambda_{\gamma_0} = \Lambda_{\gamma_1}$  a.e. on  $[0, T_0] = [0, T_1]$ .

For a grafting function  $\phi: [0, s_0] \rightarrow [0, s_1]$  and  $t \in [0, s_0]$ , define:

$$\omega^+(t) = \lim_{h \rightarrow 0^+} \phi(t+h) - \phi(t), \quad \omega^-(t) = \lim_{h \rightarrow 0^+} \phi(t) - \phi(t-h).$$

We also adopt the convention that  $\omega^+(s_0) = 0$ , while  $\omega^-(0) = \phi(0)$ . Note that the limits above exist because  $\phi$  is increasing.

**(5.5) Lemma.** *Let  $\phi: [0, s_0] \rightarrow [0, s_1]$  be a grafting function, and let  $X^\pm$  and  $\delta^\pm$  be as in definition (5.3(a)).*

- (a)  $t \in X^\pm$  if and only if  $\omega^\pm(t) > 0$ . In this case,  $\delta^\pm(t) = \omega^\pm(t)$ .
- (b)  $X^\pm$  and  $\delta^\pm$  are uniquely determined by  $\phi$ .
- (c) If  $\phi_0: [0, s_0] \rightarrow [0, s_1]$  and  $\phi_1: [0, s_1] \rightarrow [0, s_2]$  are grafting functions then so is  $\phi = \phi_1 \circ \phi_0$ . Moreover,

$$X_0^\pm \subset X^\pm \quad \text{and} \quad \delta_0^\pm \leq \delta^\pm.$$

(Here  $\delta_0^\pm$  correspond to  $\phi_0$ ,  $\delta^\pm$  correspond to  $\phi$ , and so forth.)

*Proof.* The proof will be split into parts.

- (a) Firstly,  $\omega^+(s_0) = 0$  by convention and  $s_0 \notin X^+$  because  $X^+ \subset [0, s_0)$ . Secondly,  $\omega^-(0) = \phi(0)$  by convention, and (4) tells us that  $0 \in X^-$  if and only if  $\phi(0) \neq 0$ , in which case  $\delta^-(0) = \phi(0)$ . This proves the assertion for  $t = 0$  (resp.  $t = s_0$ ) and  $X^-$  (resp.  $X^+$ ).

Since

$$\sum_{x \in X^+} \delta^+(x) + \sum_{x \in X^-} \delta^-(x) \leq s_1 - s_0,$$

given  $\varepsilon > 0$  there exist finite subsets  $F^\pm \subset X^\pm$  such that

$$\sum_{x \in X^+ \setminus F^+} \delta^+(x) + \sum_{x \in X^- \setminus F^-} \delta^-(x) < \varepsilon.$$

Suppose  $t \notin X^+$ ,  $t < s_0$ . Then there exists  $\eta$ ,  $0 < \eta < \varepsilon$ , such that  $[t, t + \eta] \cap F^+ = \emptyset$  and  $[t, t + \eta] \cap F^-$  is either empty or  $\{x\}$ . In any case,

$$\omega^+(t) \leq \phi(t + \eta) - \phi(t) < \eta + \varepsilon < 2\varepsilon,$$

which proves that  $\omega^+(t) = 0$ .

Conversely, suppose that  $t \in X^+$ . Then clearly  $\omega^+(t) \geq \delta^+(t)$ . Moreover, an argument entirely similar to the one above shows that  $\omega^+(t) \leq \delta^+(t) + 2\varepsilon$  for any  $\varepsilon > 0$ , hence  $\omega^+(t) = \delta^+(t) > 0$ . The results for  $X^-$  (and  $t > 0$ ) follow by symmetry.

- (b) Since  $\omega^\pm$  are determined by  $\phi$ , the same must be true of  $X^\pm$  and  $\delta^\pm$ , by part (a). The converse is an obvious consequence of the definition of grafting function in (4).
- (c) Let  $\phi_1, \phi_0$  be as in the statement and set  $X_i = X_i^- \cup X_i^+$ ,  $i = 0, 1$ , and  $X = X_0 \cup \phi_0^{-1}(X_1)$ . Then  $X$  is countable since both  $X_0$  and  $X_1$  are

countable and  $\phi_0$  is injective. Moreover, if  $(a, b) \subset (0, s_0) \setminus X$  then

$$\phi_1(\phi_0(t)) = \phi_1(t + c_0) = t + c_0 + c_1 \quad (t \in (a, b))$$

for some constants  $c_0, c_1 \geq 0$ . In addition,  $\phi_1 \circ \phi_0$  is increasing, as  $\phi_1$  and  $\phi_0$  are both increasing. Thus,  $\phi_1 \circ \phi_0$  is a grafting function by (5.4(e)).

For the second assertion, let  $x \in X_0^+$  and  $h > 0$  be arbitrary. Then

$$\phi_1(\phi_0(x + h)) - \phi_1(\phi_0(x)) \geq \phi_0(x + h) - \phi_0(x) \geq \omega_0^+(x),$$

hence  $\omega^+(x) \geq \omega_0^+(x) > 0$ . Similarly, if  $x \in X_0^-$  then  $\omega^-(x) \geq \omega_0^-(x) > 0$ .

Therefore, it follows from part (a) that  $X_0^\pm \subset X^\pm$  and  $\delta_0^\pm \leq \delta^\pm$ .  $\square$

**(5.6) Lemma.** *The grafting relation  $\preccurlyeq$  is a partial order over  $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ .*

*Proof.* Suppose  $\gamma_0, \gamma_1$  are as in (5.3), with  $\gamma_0 \preccurlyeq \gamma_1$  and  $\gamma_1 \preccurlyeq \gamma_0$ . Let  $\phi_0: [0, T_0] \rightarrow [0, T_1]$  and  $\phi_1: [0, T_1] \rightarrow [0, T_0]$  be the corresponding grafting functions. By (5.4(d)), the existence of such functions implies that  $T_0 = T_1$ , which, in turn, implies that  $\phi_0(t) = t = \phi_1(t)$  for all  $t$ . Hence  $\Lambda_{\gamma_0} = \Lambda_{\gamma_1} \circ \phi_0 = \Lambda_{\gamma_1}$ , and it follows that  $\gamma_0 = \gamma_1$ . This proves that  $\preccurlyeq$  is antisymmetric.

Now suppose  $\gamma_0 \preccurlyeq \gamma_1$ ,  $\gamma_1 \preccurlyeq \gamma_2$  and let  $\phi_i: [0, T_i] \rightarrow [0, T_{i+1}]$  be the corresponding grafting functions,  $i = 0, 1$ . By (5.5(c)),  $\phi = \phi_1 \circ \phi_0$  is also a grafting function. Furthermore,

$$\Lambda_{\gamma_0} = \Lambda_{\gamma_1} \circ \phi_0 = (\Lambda_{\gamma_2} \circ \phi_1) \circ \phi_0 = \Lambda_{\gamma_2} \circ \phi$$

by hypothesis, so  $\gamma_0 \preccurlyeq \gamma_2$ , proving that  $\preccurlyeq$  is transitive.

Finally, it is clear that  $\preccurlyeq$  is reflexive.  $\square$

**(5.7) Lemma.** *Let  $\Gamma = (\gamma_s)_{s \in [a, b]}$ ,  $\gamma_s \in \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ , be a chain of grafts. Then there exists a unique extension of  $\Gamma$  to a chain of grafts on  $[a, b]$ .*

*Proof.* For  $s_0 < s_1 \in [a, b]$ , let  $\phi_{s_0, s_1}: [0, s_0] \rightarrow [0, s_1]$  be the grafting function corresponding to  $\gamma_{s_0} \preccurlyeq \gamma_{s_1}$  and similarly for  $X_{s_0, s_1}^\pm, \delta_{s_0, s_1}^\pm, \omega_{s_0, s_1}^\pm$ .

Suppose  $s_0 < s_1 < s_2$ . By hypothesis,  $\phi_{s_0, s_2} = \phi_{s_1, s_2} \circ \phi_{s_0, s_1}$ . Therefore, by (5.5(c)),

$$X_{s_0, s_1}^\pm \subset X_{s_0, s_2}^\pm \quad \text{and} \quad \delta_{s_0, s_1}^\pm \leq \delta_{s_0, s_2}^\pm \quad (s_0 < s_1 < s_2). \quad (5)$$

Fix  $s_0 \in [a, b)$  and set

$$X_{s_0, b}^\pm = \bigcup_{s_0 < s < b} X_{s_0, s}^\pm \quad \text{and} \quad \delta_{s_0, b}^\pm = \sup_{s_0 < s < b} \{\delta_{s_0, s}^\pm\}.$$

Since  $(X_{s_0,s}^\pm)$  is an increasing family of countable sets,  $X_{s_0,b}^\pm$  must also be countable. Define  $\phi_{s_0,b}: [0, s_0] \rightarrow [0, b]$  by

$$\phi_{s_0,b}(t) = t + \sum_{x < t, x \in X_{s_0,b}^+} \delta_{s_0,b}^+(x) + \sum_{x \leq t, x \in X_{s_0,b}^-} \delta_{s_0,b}^-(x).$$

Then  $\phi_{s_0,b}$  is a grafting function for any  $s_0$  by construction, and for  $s_0 < s_1$  we have

$$\phi_{s_0,b} = \lim_{s \rightarrow b-} \phi_{s_0,s} = \lim_{s \rightarrow b-} \phi_{s_1,s} \circ \phi_{s_0,s_1} = \phi_{s_1,b} \circ \phi_{s_0,s_1}.$$

Before defining the curve  $\gamma_b$ , we construct its logarithmic derivative  $\Lambda$ . For each  $s < b$ , let

$$E_s = \phi_{s,b}([0, s]), \quad E = \bigcup_{s < b} E_s.$$

Then  $\mu(E_s) = s$  for all  $s$ , hence  $[0, b] \setminus E$  has measure zero, which implies that  $E$  is measurable and  $\mu(E) = b$ . (Here  $\mu$  denotes Lebesgue measure.) For  $u \in E$ ,  $u = \phi_{s,b}(t)$  for some  $t \in [0, s]$  and  $s \in [a, b)$ , set

$$\Lambda(u) = \Lambda(\phi_{s,b}(t)) = \Lambda_s(t) \quad (u \in E), \quad (6)$$

where  $\Lambda_s$  denotes the logarithmic derivative of  $\gamma_s$ . Observe that  $\Lambda$  is well-defined, for if  $\phi_{s_0,b}(t_0) = u = \phi_{s_1,b}(t_1)$ , with  $s_0 < s_1$ , then

$$\phi_{s_1,b}(t_1) = \phi_{s_0,b}(t_0) = \phi_{s_1,b} \circ \phi_{s_0,s_1}(t_0),$$

hence  $t_1 = \phi_{s_0,s_1}(t_0)$  (because  $\phi_{s_0,s_1}$  is increasing) and thus

$$\Lambda_{s_1}(t_1) = \Lambda_{s_1}(\phi_{s_0,s_1}(t_0)) = \Lambda_{s_0}(t_0).$$

Moreover, by (5.2),

$$\Lambda(u) = \begin{pmatrix} 0 & -\sin \rho(u) & 0 \\ \sin \rho(u) & 0 & -\cos \rho(u) \\ 0 & \cos \rho(u) & 0 \end{pmatrix}$$

where  $\rho(u) = \rho_{s_0}(t)$  if  $u = \phi_{s_0,b}(t)$ . The measurability of  $\rho$  follows from that of each  $\rho_s$ . Thus, the entries of  $\Lambda$  belong to  $L^2[0, b]$  and the initial value problem  $\dot{\Phi} = \Phi \Lambda$ ,  $\Phi(0) = I$ , has a unique solution  $\Phi: [0, b] \rightarrow \mathbf{SO}_3$ . Naturally, we define  $\gamma_b(t) = \Phi(t)e_1$ .

Let  $X_{s,b} = X_{s,b}^+ \cup X_{s,b}^-$  and suppose that  $(\alpha, \beta)$  is one of the intervals which form  $(0, s) \setminus X_{s,b}$ . Then  $\phi_{s,b}(\alpha, \beta) \subset E_s \subset [0, b]$  is an interval of measure

$\beta - \alpha$ ; we have  $\Lambda(t) = \Lambda_s(t - c)$  for  $t \in \phi_{s,b}(\alpha, \beta)$  and a constant  $c \geq 0$ , so that the restriction of  $\gamma_b$  to this interval is just  $\gamma_s|[\alpha, \beta]$  composed with a rotation of  $\mathbf{S}^2$ . In particular, we deduce that the geodesic curvature  $\kappa$  of  $\gamma_b$  satisfies  $\kappa_1 < \kappa < \kappa_2$  a.e. on  $\phi_s(\alpha, \beta)$ . Since  $\lim_{s \rightarrow b} \mu(E_s) = b$ , this argument shows that  $\kappa_1 < \kappa < \kappa_2$  a.e. on  $[0, b]$ . We claim also that  $\Phi(b) = Q$ . To see this, let  $\bar{\Lambda}_s: [0, b] \rightarrow \mathfrak{so}_3$  be the extension of  $\Lambda_s$  by zero to all of  $[0, b]$ . If  $\bar{\Phi}_s$  is the solution to the initial value problem  $\dot{\bar{\Phi}}_s = \bar{\Phi}_s \bar{\Lambda}_s$ ,  $\bar{\Phi}_s(0) = I$ , we have  $\Phi_s(b) = \Phi_s(s) = Q$ . Since  $\bar{\Lambda}_s$  converges to  $\Lambda$  in the  $L^2$ -norm, it follows from continuous dependence on the parameters of a differential equation that

$$|\Phi(b) - Q| = \lim_{s \rightarrow b} |\Phi(b) - \Phi_s(b)| = 0.$$

The curve  $\gamma_b$  satisfies  $\gamma_s \preceq \gamma_b$  for any  $s \leq b$  by construction. Conversely, if this condition is satisfied then (6) must hold, showing that  $\gamma_b$  is the unique curve with this property. This completes the proof.  $\square$

### Adding loops

This subsection presents adaptations of a few concepts and results contained in §5 of [12]. Let  $\kappa_0 \in \mathbf{R}$ ,  $\rho_0 = \operatorname{arccot} \kappa_0$  and  $Q \in \mathbf{SO}_3$  be fixed throughout the discussion.

For arbitrary  $\rho_1 \in (0, \rho_0)$ , define  $\sigma^{\rho_1}$  to be the unique circle in  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$  of radius of curvature  $\rho_1$ :

$$\begin{aligned} \sigma^{\rho_1}(t) &= \cos \rho_1 (\cos \rho_1, 0, \sin \rho_1) \\ &\quad + \sin \rho_1 (\sin \rho_1 \cos(2\pi t), \sin(2\pi t), -\cos \rho_1 \cos(2\pi t)), \end{aligned}$$

and let  $\sigma_n^{\rho_1} \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  be  $\sigma^{\rho_1}$  traversed  $n$  times; in symbols,  $\sigma_n^{\rho_1}(t) = \sigma^{\rho_1}(nt)$ ,  $t \in [0, 1]$ . As we have seen in (4.4), if  $\rho_1, \rho_2 < \rho_0$  then  $\sigma^{\rho_1}$  and  $\sigma^{\rho_2}$  are homotopic within  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ .

Now let  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(Q)$ ,  $n \in \mathbf{N}$ ,  $\varepsilon > 0$  be small and  $t_0 \in (0, 1)$ . Let  $\gamma^{[t_0 \# n]}$  be the curve obtained by inserting (a suitable rotation of)  $\sigma_n^{\rho_1}$  at  $\gamma(t_0)$ , as depicted in fig. 7. More explicitly,

$$\gamma^{[t_0 \# n]}(t) = \begin{cases} \gamma(t) & \text{if } 0 \leq t \leq t_0 - 2\varepsilon \\ \gamma(2t - t_0 + 2\varepsilon) & \text{if } t_0 - 2\varepsilon \leq t \leq t_0 - \varepsilon \\ \Phi_\gamma(t_0) \sigma_n^{\rho_1} \left( \frac{t - t_0 + \varepsilon}{2\varepsilon} \right) & \text{if } t_0 - \varepsilon \leq t \leq t_0 + \varepsilon \\ \gamma(2t - t_0 - 2\varepsilon) & \text{if } t_0 + \varepsilon \leq t \leq t_0 + 2\varepsilon \\ \gamma(t) & \text{if } t_0 + 2\varepsilon \leq t \leq 1 \end{cases}$$

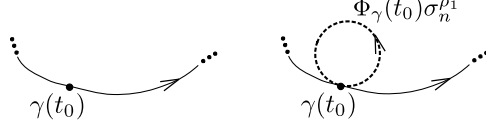


Figure 7: A curve  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(Q)$  and the curve  $\gamma^{[t_0\#n]}$  obtained from  $\gamma$  by adding loops at  $\gamma(t_0)$ .

The precise values of  $\varepsilon$  and  $\rho_1$  are not important, in the sense that different values of both parameters yield curves that are homotopic. For  $t_0 \neq t_1 \in (0, 1)$  and  $n_0, n_1 \in \mathbf{N}$ , the curve  $(\gamma^{[t_0\#n_0]})^{[t_1\#n_1]}$  will be denoted by  $\gamma^{[t_0\#n_0; t_1\#n_1]}$ .

We shall now explain how to spread loops along a curve, as in fig. 8; to do this, a special parametrization is necessary. Given  $\gamma \in \mathcal{L}_{-\infty}^{+\infty}(Q)$ , let  $\Lambda_\gamma = (\Phi_\gamma)^{-1}\dot{\Phi}_\gamma: [0, 1] \rightarrow \mathfrak{so}_3$  denote its logarithmic derivative. Since the entries of  $\Lambda_\gamma$  are  $L^2$  functions and  $[0, 1]$  is bounded,

$$M = \int_0^1 |\Lambda_\gamma(t)| dt < +\infty. \quad (7)$$

Define a function  $\tau: [0, 1] \rightarrow [0, 1]$  by

$$\tau(t) = \frac{1}{M} \int_0^t |\Lambda_\gamma(u)| du.$$

Then  $\tau$  is a monotone increasing function, hence it admits an inverse. If we reparametrize  $\gamma$  by  $\tau \mapsto \gamma(t(\tau))$ ,  $\tau \in [0, 1]$ , then its logarithmic derivative with respect to  $\tau$  satisfies

$$|\Lambda_\gamma(\tau)| = |\dot{\Phi}_\gamma(t(\tau))| \dot{t}(\tau) = |\Lambda_\gamma(t(\tau))| \frac{M}{|\Lambda_\gamma(t(\tau))|} = M.^1$$

Therefore, using (2.1), we may assume at the outset that all curves  $\gamma \in \mathcal{L}_{-\infty}^{+\infty}(Q)$  are parametrized so that  $|\dot{\Phi}_\gamma| = |\Lambda_\gamma|$  is constant (and finite). With this assumption in force, let  $n \in \mathbf{N}$ ,  $\rho_1 \in (0, \pi)$  and define a map  $F_n: \mathcal{L}_{-\infty}^{+\infty}(Q) \rightarrow \mathcal{L}_{-\infty}^{+\infty}(Q)$  by:

$$F_n(\gamma)(t) = \Phi_\gamma(t)\sigma_n^{\rho_1}(t) \quad (\gamma \in \mathcal{L}_{-\infty}^{+\infty}(Q), t \in [0, 1]). \quad (8)$$



Figure 8:

<sup>1</sup>The parameter  $\tau$  is a multiple of the curvature parameter considered in (5.1).



Using that  $\dot{\Phi}_\gamma = \Phi_\gamma \Lambda_\gamma$  (where  $\dot{\cdot}$  denotes differentiation with respect to  $t$ ), we find that

$$\dot{F}_n(\gamma) = \Phi_\gamma(\Lambda_\gamma \sigma_n^{\rho_1} + \dot{\sigma}_n^{\rho_1}), \quad (9)$$

and this allows us to conclude that  $\Phi_{F_n(\gamma)}(0) = \Phi_\gamma(0)$  and  $\Phi_{F_n(\gamma)}(1) = \Phi_\gamma(1)$  for any admissible curve  $\gamma$ , so that  $F_n$  does indeed map  $\mathcal{L}_{-\infty}^{+\infty}(Q)$  to itself. Moreover,  $F_n(\gamma)$  is never homotopic to  $F_m(\gamma)$  when  $m \not\equiv n \pmod{2}$ . This is because the two curves have different final lifted frames:  $\tilde{\Phi}_{F_n(\gamma)}(1) = (-1)^{n-m} \tilde{\Phi}_{F_m(\gamma)}(1)$  in  $\mathbf{S}^3$ .

**(5.8) Lemma.** *Let  $\kappa_0 = \cot \rho_0 \in \mathbf{R}$ ,  $Q \in \mathbf{SO}_3$ ,  $\rho_1 \in (0, \rho_0)$ ,  $K$  be compact and  $f: K \rightarrow \mathcal{L}_{-\infty}^{+\infty}(Q)$  be continuous. Then the image of  $F_n \circ f$  is contained in  $\mathcal{L}_{\kappa_0}^{+\infty}(Q)$  for all sufficiently large  $n$ .*

*Proof.* In order to simplify the notation, we will prove the lemma when  $K$  consists of a single point. The proof still works in the more general case because all that we need is a uniform bound on  $|\Lambda_{f(a)}|$  for  $a \in K$ . Denoting  $\sigma_1^{\rho_1}$  simply by  $\sigma$ , we may rewrite (9) as:

$$\dot{F}_n(\gamma)(t) = n \Phi_\gamma(t) \left( \dot{\sigma}(nt) + O\left(\frac{1}{n}\right) \right) \quad (t \in [0, 1]), \quad (10)$$

where  $O(\frac{1}{n})$  denotes a term such that  $n |O(\frac{1}{n})|$  is uniformly bounded over  $[0, 1]$  as  $n$  ranges over all of  $\mathbf{N}$ . (In this case,  $n |O(\frac{1}{n})| = |\Lambda_\gamma(t)| = M$  for all  $t \in [0, 1]$ , with  $M$  as in (7).) Therefore,

$$F_n(\gamma)(t) \times \frac{\dot{F}_n(\gamma)(t)}{|\dot{F}_n(\gamma)(t)|} = \Phi_\gamma(t) \left( \sigma(nt) \times \frac{\dot{\sigma}(nt)}{|\dot{\sigma}(nt)|} \right) + O\left(\frac{1}{n}\right). \quad (11)$$

Let  $\Phi_{F_n(\gamma)}$  (resp.  $\Phi_\sigma$ ) denote the frame of  $F_n(\gamma)$  (resp.  $\sigma$ ) and  $\Lambda_{F_n(\gamma)}$  (resp.  $\Lambda_\sigma$ ) its logarithmic derivative. It follows from (8), (10) and (11) that

$$\Phi_{F_n(\gamma)}(t) = \Phi_\gamma(t) \Phi_\sigma(nt) + O\left(\frac{1}{n}\right).$$

Differentiating both sides of this equality, we obtain that

$$\dot{\Phi}_{F_n(\gamma)}(t) = \dot{\Phi}_\gamma(t) \Phi_\sigma(nt) + n \Phi_\gamma(t) \dot{\Phi}_\sigma(nt) + O(1) = n \left( \Phi_\gamma(t) \dot{\Phi}_\sigma(nt) + O\left(\frac{1}{n}\right) \right).$$

Multiplying on the left by the inverse of  $\Phi_{F_n(\gamma)}$ , we finally conclude that

$$\Lambda_{F_n(\gamma)}(t) = n \left( \Lambda_\sigma(nt) + O\left(\frac{1}{n}\right) \right). \quad (12)$$

Recall that, by the definition of logarithmic derivative (eq. (2), §1),

$$\Lambda_{F_n(\gamma)} = \begin{pmatrix} 0 & -|\dot{F}_n(\gamma)| & 0 \\ |\dot{F}_n(\gamma)| & 0 & -|\dot{F}_n(\gamma)|\kappa_{F_n(\gamma)} \\ 0 & |\dot{F}_n(\gamma)|\kappa_{F_n(\gamma)} & 0 \end{pmatrix} \quad (13)$$

$$\text{and } \Lambda_\sigma = \begin{pmatrix} 0 & -|\dot{\sigma}| & 0 \\ |\dot{\sigma}| & 0 & -|\dot{\sigma}|\kappa_1 \\ 0 & |\dot{\sigma}|\kappa_1 & 0 \end{pmatrix}, \quad (14)$$

where  $\kappa_{F_n(\gamma)}$  (resp.  $\kappa_1 = \cot \rho_1$ ) denotes the geodesic curvature of  $F_n(\gamma)$  (resp.  $\sigma$ ). Comparing the (3,2)-entries of (12) and (13), and using (10), we deduce that

$$n(|\dot{\sigma}(nt)| + O(\frac{1}{n}))\kappa_{F_n(\gamma)}(t) = n(|\dot{\sigma}(nt)|\kappa_1 + O(\frac{1}{n})).$$

Therefore  $\lim_{n \rightarrow +\infty} \kappa_{F_n(\gamma)} = \kappa_1 > \kappa_0$  uniformly over  $[0, 1]$ , as required.  $\square$

**(5.9) Lemma.** *Let  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(Q)$ ,  $t_0 \in (0, 1)$ . Then  $\gamma^{[t_0 \# n]} \simeq F_n(\gamma)$  within  $\mathcal{L}_{\kappa_0}^{+\infty}(Q)$  for all sufficiently large  $n \in \mathbf{N}$ .*

*Proof.* Intuitively, the homotopy is obtained by pushing the loops in  $F_n(\gamma)$  towards  $\gamma(t_0)$ . If  $n$  is large enough, then we can guarantee that the curvature remains greater than  $\kappa_0$  throughout the deformation; the proof is similar to that of (5.8), so we will omit it. See lemma 5.4 in [12] for the details when  $\kappa_0 = 0$ .  $\square$

The next result states that after we add enough loops to a curve, it becomes so flexible that any condition on the curvature may be safely forgotten.

**(5.10) Lemma.** *Let  $\gamma_0, \gamma_1 \in \mathcal{L}_{\kappa_0}^{+\infty}(Q)$  be two curves in the same component of  $\mathcal{I}(Q) = \mathcal{L}_{-\infty}^{+\infty}(Q)$ . Then  $F_n(\gamma_0)$  and  $F_n(\gamma_1)$  lie in the same component of  $\mathcal{L}_{\kappa_0}^{+\infty}(Q)$  for all sufficiently large  $n \in \mathbf{N}$ .*

*Proof.* Let  $\gamma_0, \gamma_1$  be two curves in the same component of  $\mathcal{L}_{-\infty}^{+\infty}(Q)$ . Taking  $K = [0, 1]$  and  $h: K \rightarrow \mathcal{L}_{-\infty}^{+\infty}(Q)$  to be a path joining  $\gamma_0$  and  $\gamma_1$ , we conclude from (5.8) that  $g = F_n \circ h$  is a path in  $\mathcal{L}_{\kappa_0}^{+\infty}(Q)$  joining both curves if  $n$  is sufficiently large.  $\square$

Thus, if we can find a way to deform  $\gamma_i$  into  $F_{2n}(\gamma_i)$  for large  $n$ ,  $i = 0, 1$ , then the question of deciding whether  $\gamma_0$  and  $\gamma_1$  are homotopic reduces to the easy verification of whether their final lifted frames  $\tilde{\Phi}_{\gamma_0}(1)$  and  $\tilde{\Phi}_{\gamma_1}(1)$  agree. One way to achieve this is to graft arbitrarily long arcs of circles onto such a curve; this is possible if it is diffuse (see fig. 9 below).

### Grafting non-condensed curves

**(5.11) Proposition.** *Let  $\kappa_0 \in \mathbf{R}$  and suppose that  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  is diffuse. Then  $\gamma$  is homotopic to a circle traversed a number of times.*

*Proof.* Let  $\gamma: [0, T] \rightarrow \mathbf{S}^2$  be parametrized by curvature and let  $\tilde{\Lambda}: [0, T] \rightarrow \tilde{\mathfrak{so}}_3$  be its (lifted) logarithmic derivative. Since  $\gamma$  is diffuse, we can find  $0 < t_1 < t_2 < T$  and  $\rho_1, \rho_2 \in [0, \rho_0]$  such that  $C_\gamma(t_1, \rho_1) = -C_\gamma(t_2, \rho_2)$ . By deforming  $\gamma$  in a neighborhood of  $\gamma(t_2)$  if necessary, we can actually assume that  $\rho_1, \rho_2 \in (0, \rho_0)$ . Set  $z_i = \tilde{\Phi}(t_i)$ ,

$$\chi_i = C_\gamma(t_i, \rho_i) = \cos \rho_i \gamma(t_i) + \sin \rho_i \mathbf{n}(t_i) \text{ and } \lambda_i = \cos \rho_i \mathbf{i} + \sin \rho_i \mathbf{k} \quad (i = 1, 2).$$

Identifying  $\mathbf{S}^2$  with the unit imaginary quaternions, we have

$$z_i \lambda_i z_i^{-1} = \chi_i \quad (i = 1, 2). \quad (15)$$

We will define a family of curves  $s \mapsto \gamma_s$ ,  $s \geq 0$ , as follows: First, let  $\tilde{\Lambda}_s: [0, T + 2s] \rightarrow \tilde{\mathfrak{so}}_3$  be given by:

$$\tilde{\Lambda}_s(t) = \begin{cases} \tilde{\Lambda}(t) & \text{if } 0 \leq t \leq t_1 \\ \frac{1}{2}\lambda_1 & \text{if } t_1 \leq t \leq t_1 + s \\ \tilde{\Lambda}(t - s) & \text{if } t_1 + s \leq t \leq t_2 + s \\ \frac{1}{2}\lambda_2 & \text{if } t_2 + s \leq t \leq t_2 + 2s \\ \tilde{\Lambda}(t - 2s) & \text{if } t_2 + 2s \leq t \leq T + 2s \end{cases}$$

Next, let  $\Lambda_s \in \mathfrak{so}_3$  correspond to  $\tilde{\Lambda}_s \in \tilde{\mathfrak{so}}_3$  and define  $\Phi_s$  to be the unique solution to the initial value problem  $\Phi_s(0) = I$ ,  $\dot{\Phi}_s = \Phi_s \Lambda_s$ . Finally, set  $\gamma_s = \Phi_s e_1$ . Geometrically, when  $s = 2\pi k$ ,  $\gamma_s$  is obtained from  $\gamma$  by grafting a circle of radius  $\rho_1$  traversed  $k$  times at  $\gamma(t_1)$  and another circle of radius  $\rho_2$  traversed  $k$  times at  $\gamma(t_2)$  (see fig. 9). We claim that  $\gamma_s \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  for all  $s \geq 0$ .

Indeed, we have

$$\tilde{\Phi}_s(t) = \begin{cases} \tilde{\Phi}(t) & \text{if } 0 \leq t \leq t_0 \\ z_1 \exp\left(\frac{\lambda_1}{2}(t - t_1)\right) & \text{if } t_1 \leq t \leq t_1 + s \\ \exp\left(\frac{\lambda_1}{2}s\right) \tilde{\Phi}(t - s) & \text{if } t_1 + s \leq t \leq t_2 + s \\ \exp\left(\frac{\lambda_1}{2}s\right) z_2 \exp\left(\frac{\lambda_2}{2}(t - t_2 - s)\right) & \text{if } t_2 + s \leq t \leq t_2 + 2s \\ \exp\left(\frac{\lambda_1}{2}s\right) \exp\left(\frac{\lambda_2}{2}s\right) \tilde{\Phi}(t - 2s) & \text{if } t_2 + 2s \leq t \leq T + 2s \end{cases}$$

where we have used (15) to write

$$(z_1 \exp(\frac{s\lambda_1}{2}))(z_1^{-1} \tilde{\Phi}(t-s)) = \exp(\frac{s\chi_1}{2}) \tilde{\Phi}(t-s),$$

which yields the expression for  $\tilde{\Phi}(t)$  when  $t \in [t_1, t_1 + s]$ , and similarly for the interval  $[t_2 + 2s, T + 2s]$ . In particular, we deduce that the final lifted frame is:

$$\tilde{\Phi}_s(T + 2s) = \exp(\frac{s\chi_1}{2}) \exp(\frac{s\chi_2}{2}) \tilde{\Phi}(T) = \tilde{\Phi}(T),$$

as  $\chi_2 = -\chi_1$  by hypothesis. This proves that each  $\gamma_s$  has the correct final frame. The curvature  $\kappa^s$  of  $\gamma_s$  clearly satisfies  $\kappa^s > \kappa_0$  almost everywhere in  $[0, t_1] \cup [t_1 + s, t_2 + s] \cup [t_2 + 2s, T + 2s]$ , because, by construction, the restriction of  $\gamma_s$  to each of these intervals is the composition of a rotation of  $\mathbf{S}^2$  with an arc of  $\gamma$ . Moreover, the restriction of  $\gamma_s$  to the interval  $[t_1, t_1 + s]$  is an arc of circle of radius of curvature  $\rho_1 < \rho_0$ ; similarly, the restriction of  $\gamma_s$  to  $[t_2 + s, t_2 + 2s]$  is an arc of circle of radius of curvature  $\rho_2 < \rho_0$ . Therefore  $\kappa^s > \kappa_0$  almost everywhere on  $[0, T + 2s]$ , and we conclude that  $\gamma_s \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ .

We have thus proved that  $\gamma$  is homotopic to  $\gamma^{[t_0 \# n; t_1 \# n]}$  for all  $n \in \mathbf{N}$  when  $\gamma$  is diffuse. The proposition now follows from (5.9) and (5.10) combined.  $\square$

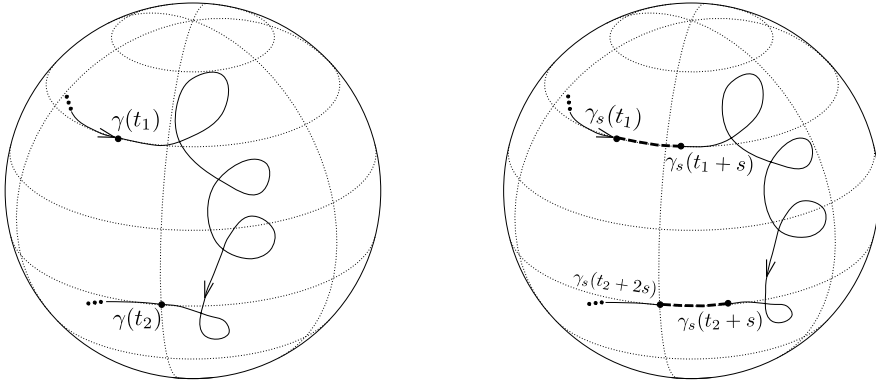


Figure 9: Grafting arcs of circles onto a diffuse curve, as described in (5.11).

The next result says that we can still graft small arcs of circle onto  $\gamma$  even when it is not diffuse, as long as it is also not condensed.

**(5.12) Proposition.** *Suppose that  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  is non-condensed. Then there exist  $\varepsilon > 0$  and a chain of grafts  $(\gamma_s)$  such that  $\gamma_0 = \gamma$ ,  $\gamma_s \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  and  $\text{tot}(\gamma_s) = \text{tot}(\gamma) + s$  for all  $s \in [0, \varepsilon]$ .*

*Proof.* (In this proof the identification of  $\mathbf{S}^2$  with the set of unit imaginary quaternions used in (5.11) is still in force.) Let  $\gamma: [0, T] \rightarrow \mathbf{S}^2$  be parametrized by curvature and let  $\tilde{\Lambda}: [0, T] \rightarrow \mathfrak{so}_3$  be its (lifted) logarithmic derivative.

Since  $\gamma$  is not condensed, 0 lies in the interior of the convex closure of the image  $C$  of  $C_\gamma$  by (11.2). Hence, by (11.5), we can find a 3-dimensional simplex with vertices in  $C$  containing 0 in its interior. In symbols, we can find  $0 < t_1 < t_2 < t_3 < t_4 < T$  and  $s_1, s_2, s_3, s_4 > 0$ ,  $s_1 + s_2 + s_3 + s_4 = 1$ , such that

$$0 = s_1\chi_1 + s_2\chi_2 + s_3\chi_3 + s_4\chi_4, \quad (16)$$

where  $\chi_i = C_\gamma(t_i, \rho_i)$ , for some  $\rho_i \in (0, \rho_0)$ , and the  $\chi_i$  are in general position. Furthermore, these numbers  $s_i$  are the only ones which have these properties (for this choice of the  $\chi_i$ ). Define a function  $G: \mathbf{R}^4 \rightarrow \mathbf{S}^3$  by

$$G(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \exp\left(\frac{\sigma_1\chi_1}{2}\right) \exp\left(\frac{\sigma_2\chi_2}{2}\right) \exp\left(\frac{\sigma_3\chi_3}{2}\right) \exp\left(\frac{\sigma_4\chi_4}{2}\right).$$

Then  $G(0, 0, 0, 0) = \mathbf{1}$  and

$$DG_{(0,0,0,0)}(a, b, c, d) = \frac{1}{2}(a\chi_1 + b\chi_2 + c\chi_3 + d\chi_4).$$

Since the  $\chi_i$  are in general position by hypothesis, we can invoke the implicit function theorem to find some  $\delta > 0$  and, without loss of generality, functions  $\bar{\sigma}_2, \bar{\sigma}_3, \bar{\sigma}_4: (-\delta, \delta) \rightarrow \mathbf{R}$  of  $\sigma_1$  such that

$$G(\sigma_1, \bar{\sigma}_2(\sigma_1), \bar{\sigma}_3(\sigma_1), \bar{\sigma}_4(\sigma_1)) = \mathbf{1} \quad (\sigma_1 \in (-\delta, \delta)).$$

Differentiating the previous equality with respect to  $\sigma_1$  at 0 and comparing (16) we deduce that

$$\bar{\sigma}_i'(0) = \frac{s_i}{2s_1} > 0 \quad (i = 2, 3, 4).$$

Let  $s(\sigma_1) = \sigma_1 + \bar{\sigma}_2(\sigma_1) + \bar{\sigma}_3(\sigma_1) + \bar{\sigma}_4(\sigma_1)$ . Then  $s'(\sigma_1) > 0$ , hence we can write  $\sigma_1, \sigma_2, \sigma_3$  and  $\sigma_4$  as a function of  $s$  in a neighborhood of 0. The conclusion is thus that there exist  $\varepsilon > 0$  and non-negative functions  $\sigma_1, \sigma_2, \sigma_3, \sigma_4$  of  $s$  such that  $\sigma_1(s) + \sigma_2(s) + \sigma_3(s) + \sigma_4(s) = s$  and

$$\exp\left(\frac{\sigma_1\chi_1}{2}\right) \exp\left(\frac{\sigma_2\chi_2}{2}\right) \exp\left(\frac{\sigma_3\chi_3}{2}\right) \exp\left(\frac{\sigma_4\chi_4}{2}\right) = \mathbf{1} \quad \text{for all } s \in [0, +\varepsilon).$$

We will now use these functions to obtain  $\gamma_s, s \in [0, +\varepsilon)$ .

Define  $\tilde{\Lambda}_s: [0, T + s] \rightarrow \mathfrak{so}_3$  by:

$$\tilde{\Lambda}_s(t) = \begin{cases} \tilde{\Lambda}(t) & \text{if } 0 \leq t \leq t_1 \\ \frac{1}{2}\lambda_1 & \text{if } t_1 \leq t \leq t_1 + \sigma_1 \\ \tilde{\Lambda}(t - \sigma_1) & \text{if } t_1 + \sigma_1 \leq t \leq t_2 + \sigma_1 \\ \frac{1}{2}\lambda_2 & \text{if } t_2 + \sigma_1 \leq t \leq t_2 + \sigma_1 + \sigma_2 \\ \tilde{\Lambda}(t - \sigma_1 - \sigma_2) & \text{if } t_2 + \sigma_1 + \sigma_2 \leq t \leq t_3 + \sigma_1 + \sigma_2 \\ \frac{1}{2}\lambda_3 & \text{if } t_3 + \sigma_1 + \sigma_2 \leq t \leq t_3 + \sigma_1 + \sigma_2 + \sigma_3 \\ \tilde{\Lambda}(t - \sigma_1 - \sigma_2 - \sigma_3) & \text{if } t_3 + \sigma_1 + \sigma_2 + \sigma_3 \leq t \leq t_4 + \sigma_1 + \sigma_2 + \sigma_3 \\ \frac{1}{2}\lambda_4 & \text{if } t_4 + \sigma_1 + \sigma_2 + \sigma_3 \leq t \leq t_4 + s \\ \tilde{\Lambda}(t - s) & \text{if } t_4 + s \leq t \leq T + s \end{cases}$$

where  $\sigma_i = \sigma_i(s)$  ( $i = 1, 2, 3, 4$ ) are the functions obtained above. Let  $\tilde{\Phi}_s: [0, T + s] \rightarrow \mathbf{S}^3$  be the solution to the initial value problem  $\tilde{\Phi}' = \tilde{\Phi}\tilde{\Lambda}$ ,  $\tilde{\Phi}(0) = \mathbf{1}$  and let  $\Phi: [0, T + s] \rightarrow \mathbf{SO}_3$  be its projection. Then using the relation  $\chi_i = z_i \lambda_i z_i^{-1}$  one finds by a verification entirely similar to the one in the proof of (5.11) that

$$\tilde{\Phi}_s(T + s) = \exp\left(\frac{\sigma_1 \chi_1}{2}\right) \exp\left(\frac{\sigma_2 \chi_2}{2}\right) \exp\left(\frac{\sigma_3 \chi_3}{2}\right) \exp\left(\frac{\sigma_4 \chi_4}{2}\right) \Phi(T) = \tilde{\Phi}(T).$$

Hence, each  $\gamma_s = \Phi_s e_1$  has the correct final frame. In addition, over each of the subintervals of  $[0, T + s]$  listed above,  $\gamma_s$  is either the composition of a rotation of  $\mathbf{S}^2$  with an arc of  $\gamma$ , or an arc of circle of radius  $\rho_i \in (0, \rho_0)$  ( $i = 1, 2, 3, 4$ ). We conclude from this that the geodesic curvature  $\kappa^s$  of  $\gamma_s$  satisfies  $\kappa^s > \kappa_0$  almost everywhere on  $[0, T + s]$ , that is,  $\gamma_s \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  as we wished. Finally,

$$\text{tot}(\gamma_s) = T + s = \text{tot}(\gamma) + s$$

because  $\gamma_s$  is parametrized by curvature (see (5.2)), and  $(\gamma_s)$  is a chain of grafts by construction.  $\square$

## 6

### Condensed Curves

The *rotation number*  $N(\eta)$  of a regular closed plane curve  $\eta: [0, 1] \rightarrow \mathbf{R}^2$  is simply the degree of its unit tangent vector  $\mathbf{t}: \mathbf{S}^1 \rightarrow \mathbf{S}^1$  (we may consider  $\gamma$  and  $\mathbf{t}$  to be defined on  $\mathbf{S}^1$  since  $\gamma$  is closed). Suppose now that  $\eta: [0, L] \rightarrow \mathbf{R}^2$  is parametrized by arc-length, and write

$$\mathbf{t}(s) = \exp(i\theta(s)),$$

for some angle-function  $\theta: [0, L] \rightarrow \mathbf{R}$ . Then the curvature  $\kappa$  of  $\eta$  is given by

$$\kappa(s) = \theta'(s); \quad (1)$$

furthermore, the rotation number  $N(\eta)$  of  $\eta$  is given by  $2\pi N(\eta) = \theta(L) - \theta(0)$ . These facts are explained in any textbook on differential geometry. The Whitney-Graustein theorem ([17], thm. 1) states that two regular closed plane curves are homotopic through regular closed curves if and only if they have the same rotation number.

Now suppose  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  has image contained in some closed hemisphere. Let  $h_\gamma$  be the barycenter, on  $\mathbf{S}^2$ , of the set of closed hemispheres which contain  $\text{Im}(\gamma)$  (cf. (4.11)), and let  $\text{pr}: \mathbf{S}^2 \rightarrow \mathbf{R}^2$  denote stereographic projection from  $-h_\gamma$ . Define the *rotation number*  $\nu(\gamma)$  of  $\gamma$  by  $\nu(\gamma) = -N(\eta)$ , where  $\eta = \text{pr} \circ \gamma$ . Recall that a curve  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  is called *condensed* if the image  $C$  of its caustic band  $C_\gamma: [0, 1] \times [0, \rho_0] \rightarrow \mathbf{S}^2$  is contained in some closed hemisphere. Because  $C_\gamma(t, 0) = \gamma(t)$ , any condensed curve is contained in a closed hemisphere, hence we may speak of its rotation number.

*Remark.* It is natural to ask why this notion of rotation number is not extended to a larger class of curves. For instance, if  $\gamma$  is any admissible curve then, by Sard's theorem, there exists some point  $p \in \mathbf{S}^2$  not in the image of  $\gamma$ . We could use stereographic projection from  $p$  to define the rotation number of  $\gamma$ . The trouble is that it is not clear how  $p$  can be chosen so that the resulting number is locally constant (i.e., continuous) on  $\mathcal{L}_{\kappa_0}^{+\infty}$ : A different choice of  $p$  yields a different rotation number (although its parity remains the same). In fact, the class of spherical curves for which a meaningful notion of rotation

number exists must be restricted, since it is always possible to deform a circle traversed  $\nu$  times into a circle traversed  $\nu + 2$  times in  $\mathcal{L}_{\kappa_0}^{+\infty}$  if  $\nu$  is sufficiently large.

**(6.1) Proposition.** *Let  $A$  be a connected compact space,  $\kappa_0 > 0$  and  $f: A \rightarrow \mathcal{L}_{\kappa_0}^{+\infty}(I)$  be such that  $f(a)$  is condensed for all  $a \in A$ . Then there exists  $\nu \in \mathbf{N}$  such that  $f$  is homotopic in  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$  to the constant map  $a \mapsto \sigma_\nu$ ,  $\sigma_\nu$  a circle traversed  $\nu$  times.*

The idea of the proof is to use Möbius transformations to make the curves  $\eta_a = f(a)$  so small that they become approximately plane curves. The hypothesis that the curves are condensed guarantees that the geodesic curvature does not decrease during the deformation. A slight variation of the Whitney-Graustein theorem is then used to deform the curves to a circle traversed  $\nu$  times, where  $\nu$  is the common rotation number of the curves.

We will also need the following technical result, which is a corollary of the proof of (6.1).

**(6.2) Corollary.** *Let  $\kappa_0 > 0$  and  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  be a condensed curve. Then there exists a homotopy  $s \mapsto \gamma_s \in \mathcal{L}_{\kappa_0}^{+\infty}$  ( $s \in [0, 1]$ ) such that  $\gamma_1 = \gamma$ ,  $\gamma_0$  is a parametrized circle and  $\text{Im}(C_{\gamma_s})$  is contained in an open hemisphere for each  $s \in [0, 1)$ .*

We start by defining spaces of closed curves in  $\mathbf{R}^2$  which are analogous to the spaces  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  of curves on  $\mathbf{S}^2$ .<sup>1</sup> Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ . A  $(\kappa_1, \kappa_2)$ -admissible plane curve is an element  $(c, z, \hat{v}, \hat{w})$  of  $\mathbf{R}^2 \times \mathbf{S}^1 \times L^2[0, 1] \times L^2[0, 1]$ . With such a 4-tuple we associate the unique curve  $\gamma: [0, 1] \rightarrow \mathbf{R}^2$  satisfying

$$\gamma(t) = c + \int_0^t v(\tau) \mathbf{t}(\tau) d\tau, \quad \mathbf{t}(0) = z, \quad \mathbf{t}'(t) = w(t) i \mathbf{t}(t) \quad (t \in [0, 1]),$$

where  $v$  and  $w$  are given by eq. (6) on p. 17 and  $i = (0, 1)$  is the imaginary unit. The space of all  $(\kappa_1, \kappa_2)$ -admissible plane curves is thus given the structure of a Hilbert manifold, and we define  $\mathcal{W}_{\kappa_1}^{\kappa_2}$  to be its subspace consisting of all closed curves.

Although  $\dot{\gamma}$  is defined only almost everywhere for a curve  $\gamma \in \mathcal{W}_{\kappa_1}^{\kappa_2}$ , its unit tangent vector  $\mathbf{t}$  is defined over all of  $[0, 1]$ , and if we parametrize  $\gamma$  by a multiple of arc-length instead, then  $\dot{\gamma}$  is defined and nonzero everywhere. More importantly, since  $\mathbf{t}$  is (absolutely) continuous, we may speak of the rotation number of  $\gamma$  and (1) still holds a.e..

<sup>1</sup>These spaces of plane curves will only be considered in this section.



**(6.3) Lemma.** *Let  $A$  be compact and connected,  $\kappa_0 \geq 0$  and  $A \rightarrow \mathcal{W}_{\kappa_0}^{+\infty}$ ,  $a \mapsto \eta_a$ , be a continuous map. Then there exists a homotopy  $[0, 1] \times A \rightarrow \mathcal{W}_{\kappa_0}^{+\infty}$ ,  $(s, a) \mapsto \eta_a^s$ , such that  $\eta_a^0 = \eta_a$  and*

$$\eta_a^1(t) = \sigma_N(t + t_a) \quad \text{for all } a \in A, t \in [0, 1],$$

where  $\sigma_N(t) = R_0 \exp(2\pi i N t)$  is a circle traversed  $N > 0$  times. In addition, if the image of  $\eta_a$  is contained in some ball  $B(0; R)$  for all  $a \in A$ , then we can arrange that  $\eta_a^s$  have the same property for all  $s \in [0, 1]$  and  $a \in A$ .

Thus, given a family of curves in  $\mathcal{W}_{\kappa_0}^{+\infty}$  indexed by a compact connected set, we may deform all of them to the same parametrized circle  $\sigma_N$ , except for the starting point of the parametrization.

*Proof.* Since  $A$  is connected, all the curves  $\eta_a$  have the same rotation number  $N$ . Moreover,  $N > 0$  because of (1) and the fact that  $\kappa_0 \geq 0$ .

For  $\eta \in \mathcal{W}_{\kappa_0}^{+\infty}$ , let  $z_\eta = \mathbf{t}_\eta(0)$ , where  $\mathbf{t}_\eta$  is the unit tangent vector to  $\eta$ . The homotopy  $g: [0, 1] \times A \rightarrow \mathcal{W}_{\kappa_0}^{+\infty}$  by translations,

$$g(s, a)(t) = \eta_a(t) - s(iz_{\eta_a} + \eta_a(0)) \quad (s, t \in [0, 1], a \in A),$$

preserves the curvature and, for any  $a \in A$ ,  $g(1, a)$  has the property that it starts at some  $z \in \mathbf{S}^1$  in the direction  $iz$ . Thus, we may assume without loss of generality that the original curves  $\eta_a$  have this property.

Let  $\rho_0 = \frac{1}{\kappa_0}$ ,  $L(\eta_a)$  denote the length of  $\eta_a$ ,  $L_0 = \min_{a \in A} \{L(\eta_a)\}$  and let  $R_1 > 0$  satisfy

$$R_1 < \min \left\{ \frac{L_0}{2\pi N}, \rho_0 \right\}^2. \quad (2)$$

Define  $f: [0, 1] \times A \rightarrow \mathcal{W}_{\kappa_0}^{+\infty}$  to be the homotopy given by

$$f(s, a)(t) = \eta_a(0) + \left( (1-s) + s \frac{2\pi N R_1}{L(\eta_a)} \right) (\eta_a(t) - \eta_a(0)) \quad (s, t \in [0, 1], a \in A).$$

Then  $f(1, a)$  has length  $L = 2\pi N R_1$  for all  $a \in A$ . In addition, the curvature of  $f(s, a)$  is bounded from below by  $\kappa_0$  for all  $s \in [0, 1]$ ,  $a \in A$  and almost every  $t \in [0, 1]$ , as an easy calculation using (2) shows.

The conclusion is that we lose no generality in assuming that the curves  $\eta_a$  all have the same length  $L = 2\pi N R_0$ . Further, by (2.1), we can assume that they are all parametrized by a multiple of arc-length. This implies that  $\dot{\eta}_a$  takes values on the circle  $L\mathbf{S}^1$  of radius  $L$ . Using angle-functions  $\theta_a$  with

<sup>2</sup>If  $\kappa_0 = 0$  then we adopt the convention that  $\rho_0 = +\infty$ .

$\theta_a(0) = 0$  and  $\theta_a(1) = 2\pi N$ , we can write:

$$\dot{\eta}_a(t) = Lz_a \exp(i\theta_a(t)) \quad (t \in [0, 1]),$$

where  $z_a = \mathbf{t}_{\eta_a}(0)$ . Let  $\theta(t) = 2\pi Nt$ ,  $t \in [0, 1]$ , and define

$$\theta_a^s(t) = (1-s)\theta_a(t) + s\theta(t), \quad \bar{\tau}_a^s(t) = Lz_a \exp(i\theta_a^s(t)) \quad (s, t \in [0, 1], a \in A).$$

Then  $\theta_a^s(0) = 0$  and  $\theta_a^s(1) = 2\pi N$  for all  $s \in [0, 1]$ ,  $a \in A$ . The idea is that  $\bar{\tau}_a^s$  should be the tangent vector to a curve; the problem is that this curve need not be closed. We can fix this by defining instead

$$\tau_a^s(t) = \bar{\tau}_a^s(t) - \int_0^1 \bar{\tau}_a^s(v) dv, \quad \eta_a^s(t) = -iz_a + \int_0^t \tau_a^s(v) dv.$$

The conditions  $\int_0^1 \tau_a^s(t) dt = 0$  and  $\tau_a^s(0) = \tau_a^s(1)$  then guarantee that  $\eta_a^s$  is a closed curve. Because  $\theta_a^s(1) = 2\pi N$  and  $N > 0$ ,  $\bar{\tau}_a^s$  must traverse all of  $L\mathbf{S}^1$ , so that  $\int_0^1 \bar{\tau}_a^s(v) dv$  lies in the interior of the disk bounded by this circle for any  $s \in [0, 1]$ ,  $a \in A$ . Consequently,  $\tau_a^s(t)$  never vanishes. Moreover,

$$\eta_a^0 = \eta_a \quad \text{and} \quad \eta_a^1(t) = -iz_{\eta_a} \exp(2\pi Nit) \quad \text{for all } a \in A.$$

Finally,  $\eta_a^s$  has positive curvature for all  $s \in [0, 1]$  and  $a \in A$ . Although it is easier to see this using a geometrical argument, the following computation suffices: The curvature  $\kappa_a^s$  of  $\eta_a^s$  is given by

$$\kappa_a^s(t) = \frac{\det(\tau_a^s(t), \dot{\tau}_a^s(t))}{|\tau_a^s(t)|^3} = \frac{L^2 \dot{\theta}_a^s(t)}{|\tau_a^s(t)|^3} \left[ 1 - \det \left( \int_0^1 \exp(i\theta_a^s(v)) dv, i \exp(i\theta_a^s(t)) \right) \right].$$

Because  $\theta_a^s = (1-s)\theta_a + s\theta$  is monotone increasing (recall that  $\theta'_a = \kappa_a > \kappa_0 \geq 0$  a.e. by hypothesis), the map  $t \mapsto \exp(i\theta_a^s(t))$  runs over all of  $\mathbf{S}^1$  for any  $s$  and  $a$ . As a consequence, the integral above has norm strictly less than 1, hence so does the determinant. In fact, since  $A$  is compact, we can find a constant  $C > 0$ , independent of  $a$  and  $s$ , such that

$$\kappa_a^s > C\kappa_0.$$

For  $\lambda > 0$  and an admissible plane curve  $\gamma$ , the curve  $\lambda\gamma$  has curvature given by  $\frac{\kappa}{\lambda}$ , where  $\kappa$  is the curvature of  $\gamma$ . Again using compactness of  $A$ , we may find a smooth function  $\lambda: [0, 1] \rightarrow (0, 1]$  such that  $\lambda(0) = 1$  and  $\lambda(s)$  is as small as necessary for  $s \in (0, 1]$  to guarantee that  $\kappa_a^s > \kappa_0$  for all  $s \in [0, 1]$  and  $a \in A$  if we replace  $\eta_a^s$  by  $\lambda(s)\eta_a^s$ . In addition, we can choose  $\lambda$  so that the

image of  $\lambda(s)\eta_a^s$  is contained in the ball  $B_R(0)$  if this is the case for each  $\eta_a$ . This establishes the lemma with  $R_0 = \lambda(1)$ .  $\square$

The next result states that the geodesic curvature of a curve  $\gamma: [0, 1] \rightarrow \mathbf{S}^2$  and the curvature of the plane curve obtained by projecting  $\gamma$  orthogonally on  $T_p\mathbf{S}^2$  are roughly the same, as long as the curve is contained in a small neighborhood of  $p$ .

**(6.4) Lemma.** *Let  $\kappa_0 < \kappa_1 < \kappa_2$  and  $p \in \mathbf{S}^2$  be given. Identifying  $T_p\mathbf{S}^2$  with  $\mathbf{R}^2$ , with  $p$  corresponding to the origin, let  $P: \mathbf{S}^2 \rightarrow \mathbf{R}^2$  be the orthogonal projection. Then there exists  $\varepsilon > 0$  such that:*

- (a) *If  $\gamma \in \mathcal{L}_{\kappa_2}^{+\infty}$  satisfies  $d(\gamma(t), p) < \varepsilon$  for all  $t \in [0, 1]$ , then  $\eta = P \circ \gamma \in \mathcal{W}_{\kappa_1}^{+\infty}$ .*
- (b) *If  $\eta \in \mathcal{W}_{\kappa_1}^{+\infty}$  satisfies  $|\eta(t)| < \varepsilon$  for all  $t \in [0, 1]$ , then  $\gamma = P^{-1} \circ \eta \in \mathcal{L}_{\kappa_0}^{+\infty}$ .*

In part (a),  $d$  denotes the distance function on  $\mathbf{S}^2$  and the transformation  $P^{-1}$  in part (b) is to be understood as the inverse of  $P$  when restricted to the hemisphere  $\{q \in \mathbf{S}^2 : \langle q, p \rangle > 0\}$ .

*Proof.* Since the subset of smooth curves is dense in the space of all admissible (plane or spherical) curves, it suffices to prove the lemma for  $C^2$  curves. Let  $\gamma \in \mathcal{L}_{\kappa_2}^{+\infty}$  be a  $C^2$  curve such that  $d(\gamma(t), p) < \varepsilon$  for all  $t \in [0, 1]$ . If  $0 < \varepsilon < \frac{\pi}{2}$  then  $\eta$  will also be a  $C^2$  regular curve. Let  $UT\mathbf{S}^2$  denote the unit tangent bundle of  $\mathbf{S}^2$  and  $U \subset UT\mathbf{S}^2$  the open set consisting of all vectors in the fibers of those  $q \in \mathbf{S}^2$  with  $d(p, q) < \frac{\pi}{2}$  ( $d$  being the distance on  $\mathbf{S}^2$ ). Define  $f, g: U \rightarrow \mathbf{R}$  by

$$f(u) = \frac{\det(P(u), P(q \times u))}{|P(u)|^3} \quad \text{and} \quad g(u) = \frac{\det(P(u), P(q))}{|P(u)|^3} \quad \text{for } u \in T_q\mathbf{S}^2,$$

where  $\times$  denotes the vector product in  $\mathbf{R}^3$ .

Note that we may identify  $P$  with its derivative  $dP_q: T_q\mathbf{S}^2 \rightarrow \mathbf{R}^2$  at any  $q \in \mathbf{S}^2$ , because  $P$  is the restriction of a linear transformation  $\mathbf{R}^3 \rightarrow \mathbf{R}^2$ . With this observation in mind, a straightforward calculation yields the following expression for the curvature  $\kappa_\eta$  of  $\eta = P \circ \gamma$ :

$$\kappa_\eta(t) = f(\mathbf{t}(t))\kappa_\gamma(t) - g(\mathbf{t}(t)) \quad (t \in [0, 1]).$$

Here  $\mathbf{t}$  is the unit tangent vector to  $\gamma$ .

Since  $f, g$  are continuous and  $f(u) = 1, g(u) = 0$  for all unit vectors  $u \in T_p\mathbf{S}^2$ , it follows that there exists  $\varepsilon$  such that if  $d(p, q) < \varepsilon$ , then

$$\kappa_0 < f(v)\kappa_1 - g(v) \quad \text{for any } v \in T_q\mathbf{S}^2, |v| = 1.$$

Hence, if  $d(\gamma(t), p) < \varepsilon$  for all  $t \in [0, 1]$  then  $\kappa_\eta$  satisfies the conclusion of (a).

A similar reasoning shows that, by reducing  $\varepsilon$  if necessary, we can also arrange for (b) to hold.  $\square$

*Remark.* An analogous result to (6.4), with a similar proof, holds for upper bounds on the curvature, or even lower and upper bounds simultaneously. However, since we need neither of these versions, we will not formulate them carefully.

**(6.5) Lemma.** *Let  $h \in \mathbf{S}^2$ ,  $H = \{q \in \mathbf{S}^2 : \langle q, h \rangle \geq 0\}$ , let  $\text{pr}: \mathbf{S}^2 \rightarrow \mathbf{R}^2$  denote stereographic projection from  $-h$ . Let  $\kappa_0 > 0$  and  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  be such that  $\text{Im}(C_\gamma) \subset H$ . Define  $T_r: \mathbf{S}^2 \rightarrow \mathbf{S}^2$  to be the Möbius transformation (dilatation) given by*

$$T_r(p) = \text{pr}^{-1}(r \text{pr}(p)) \quad (r \in (0, 1], p \in \mathbf{S}^2).$$

*Then, given  $\kappa_1 > \kappa_0$ , there exists  $r_0 > 0$ , depending only on  $\kappa_0$  and  $\kappa_1$ , such that the geodesic curvature  $\kappa^r$  of  $T_r(\gamma)$  satisfies  $\kappa^r > \kappa_1$  a.e. for any  $r \in (0, r_0)$ .*

*Proof.* Suppose that  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  is parametrized by its arc-length and let  $\sigma$  be a parametrization, also by arc-length, of an arc of the osculating circle to  $\gamma$  at  $\gamma(s_0)$ , i.e., let  $\sigma$  satisfy:

$$\sigma(s_0) = \gamma(s_0), \quad \sigma'(s_0) = \gamma'(s_0), \quad \sigma''(s_0) = \gamma''(s_0).$$

(It makes sense to speak of  $\gamma''$  (as an  $L^2$  map) because  $\gamma' = \mathbf{t}$  is  $H^1$  by hypothesis.) Then  $T_r \circ \sigma$  has contact of order 3 with  $T_r \circ \gamma$  at  $s_0$ , hence their geodesic curvatures at the corresponding point agree. Therefore, it suffices to prove the result for a circle  $\Sigma$  whose center  $\chi$  lies in  $H$ . Let  $\rho_i = \text{arccot } \kappa_i$ ,  $i = 0, 1$ , and  $\rho$  be the radius of curvature of  $\Sigma$ ,  $\rho < \rho_0 < \frac{\pi}{2}$ . If  $d$  denotes the distance function on  $\mathbf{S}^2$ , then  $\Sigma \subset B_d(h; \frac{\pi}{2} + \rho_0)$  (where the latter denotes the set of  $q \in \mathbf{S}^2$  such that  $d(h, q) < \frac{\pi}{2} + \rho_0$ ). Choose  $r_0$  such that

$$T_r(B_d(h; \frac{\pi}{2} + \rho_0)) \subset B_d(h; \rho_1) \text{ for all } r \in (0, r_0).$$

Then  $T_r(\Sigma)$  is a circle, for a Möbius transformation such as  $T_r$  maps circles to circles, and its diameter is at most  $2\rho_1$ . Thus, its geodesic curvature must be greater than  $\kappa_1$ . Moreover, it is clear that the choice of  $r_0$  does not depend on  $h$  or on  $\Sigma$ .  $\square$

*Proof of (6.1).* Let  $\gamma_a$  denote  $f(a)$  and let  $h_a$  be the barycenter of the set of closed hemispheres which contain  $\text{Im}(C_{\gamma_a})$ ; by (4.10), the map  $h: A \rightarrow \mathbf{S}^2$  so defined is continuous.

Let  $\text{pr}_a$  denote stereographic projection  $\mathbf{S}^2 \rightarrow \mathbf{R}^2$  from  $-h_a$ , so that  $h_a$  is projected to the origin, and define a family  $T_a^s: \mathbf{S}^2 \rightarrow \mathbf{S}^2$  of Möbius transformations by:

$$T_a^s(q) = \text{pr}_a^{-1}(s \text{pr}_a(q)) \quad (q \in \mathbf{S}^2, s \in (0, 1], a \in A).$$

Set  $\gamma_a^s = T_a^s \gamma_a$ . From (6.5) it follows that we can choose  $\delta > 0$  so small that the geodesic curvature of  $\gamma_a^\delta$  is greater than  $\kappa_0 + 2$  a.e. for any  $a \in A$ .

Now choose  $\varepsilon > 0$  as in (6.4), with  $\kappa_1 = \kappa_0 + 1$ ,  $\kappa_2 = \kappa_0 + 2$ . By reducing  $\delta$  if necessary, we can guarantee that the curves  $\gamma_a^\delta$  have image contained in  $B_d(h_a; \varepsilon)$ , for each  $a$ . Let  $\eta_a$  be the orthogonal projection of  $\gamma_a^\delta$  onto  $T_{h_a} \mathbf{S}^2$ . We are then in the setting of (6.3). The conclusion is that we can deform all  $\eta_a$  to a single circle  $\sigma_\nu$ , modulo the starting point of the parametrization, in such a way that the curves have image contained in  $B(0; \varepsilon)$  and curvature greater than  $\kappa_0 + 1$  throughout the deformation. By (6.4) again, when we project this homotopy back to  $\mathbf{S}^2$ , the geodesic curvature of the curves is always greater than  $\kappa_0$ .

To sum up, we have described a homotopy  $H: [0, 1] \times A \rightarrow \mathbf{S}^2$  such that  $H(0, a) = \gamma_a$  and  $H(1, a)$  is a circle traversed  $\nu$  times for all  $a \in A$ ; further, the geodesic curvature  $\kappa_a^s$  of  $H(s, a)$  satisfies  $\kappa_a^s(t) > \kappa_0$  for all  $s, t \in [0, 1]$ . These curves  $H(a, s)$  do not satisfy  $\Phi(0) = I = \Phi(1)$ , but we can correct this by setting

$$\bar{H}(s, a) = \Phi_{H(a, s)}(0)^{-1} H(a, s)$$

and using  $\bar{H}$  instead; this has no effect on the geodesic curvature and finishes the proof that  $f$  is null-homotopic, since  $\bar{H}(1, a)$  is the same parametrized circle for all  $a$ .  $\square$

We now provide a proof of (6.2). This result will be used to show that a notion of rotation number for non-diffuse curves, which will be introduced in the next section, coincides with the one presented at the beginning of this section.

*Proof of (6.2).* Let  $h_\gamma$  be the barycenter on  $\mathbf{S}^2$  of the set of closed hemispheres which contain  $\text{Im}(C_\gamma)$  and, as in the proof of (6.1), define  $\gamma_s = T^s \circ \gamma$ , where

$$T^s(q) = \text{pr}^{-1}(s \text{pr}(q)) \quad (q \in \mathbf{S}^2, s \in (0, 1]) \quad (3)$$

and  $\text{pr}$  denotes stereographic projection from  $-h_\gamma$ . Let  $H = \{p \in \mathbf{S}^2 : \langle p, h_\gamma \rangle > 0\}$ . We claim that  $\text{Im}(C_{\gamma_s}) \subset H$  for all  $s \in (0, 1)$ . This follows from the following two assertions:

- (i) If  $\text{Im}(C_{\gamma_s}) \subset \bar{H}$ , then there exists  $\varepsilon > 0$  such that  $\text{Im}(C_{\gamma_\sigma}) \subset H$  for all  $\sigma \in (s - \varepsilon, s)$ ;
- (ii) If  $\text{Im}(C_{\gamma_s}) \not\subset H$ , then there exists  $\varepsilon > 0$  such that  $\text{Im}(C_{\gamma_\sigma}) \not\subset \bar{H}$  for all  $\sigma \in (s, s + \varepsilon)$ .

For any  $s$ , the boundary of  $\text{Im}(C_{\gamma_s})$  is contained in the union of the images of  $\gamma_s = C_{\gamma_s}(\cdot, 0)$  and  $\check{\gamma}_s = C_{\gamma_s}(\cdot, \rho_0)$ . Moreover,  $\gamma$  has positive geodesic curvature by hypothesis, and a straightforward calculation shows that  $\check{\gamma}$  also does (the detailed calculations may be found in (8.6)).

If  $\text{Im}(C_{\gamma_s}) \subset H$  then (i) is obviously true, since  $H$  is an open hemisphere; similarly, (ii) clearly holds if  $\text{Im}(C_{\gamma_s}) \not\subset \bar{H}$ . Suppose then that  $\text{Im}(C_{\gamma_s}) \subset \bar{H}$ , but  $\text{Im}(C_{\gamma_s}) \not\subset H$  for some  $s > 0$ . This means that there exists  $t_0 \in [0, 1]$  such that either  $\gamma_s$  or  $\check{\gamma}_s$  is tangent to  $\partial H$  at  $\gamma_s(t_0)$  or  $\check{\gamma}_s(t_0)$ , respectively. In the first case,  $\mathbf{n}_{\gamma_s}(t_0) = h_\gamma$ , and in the second  $\mathbf{n}_{\gamma_s}(t_0) = -h_\gamma$ . In either case,  $C_{\gamma_s}(\{t_0\} \times [0, \rho_0])$  is an arc of the geodesic through  $\gamma_s(t_0)$  and  $h_\gamma$ . Such geodesics through  $h_\gamma$  are mapped to lines through the origin by  $\text{pr}$ , hence (3) implies that there exists  $\varepsilon > 0$  such that  $C_\gamma(t, \sigma) \subset H$  for any  $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$  and  $\sigma \in (s - \varepsilon, s)$  and  $C_\gamma(t_0, \sigma) \not\subset \bar{H}$  for any  $\sigma \in (s, s + \varepsilon)$ . Furthermore, since the geodesic curvatures of  $\gamma$ ,  $\check{\gamma}$  are positive and  $\partial H$  is a geodesic, the set of  $t_0 \in [0, 1]$  where  $\gamma$ ,  $\check{\gamma}$  are tangent to  $\partial H$  must be finite. This implies (i) and (ii).

Now let  $S = \{s \in (0, 1) : \text{Im}(C_{\gamma_s}) \not\subset H\}$ . Assume that  $S \neq \emptyset$  and let  $s_0 = \sup S$ . Applying (i) to  $\gamma_1 = \gamma$  we conclude that there exists  $\varepsilon > 0$  with  $S \cap (1 - \varepsilon, 1) = \emptyset$ . Hence,  $s_0 < 1$  and  $\text{Im}(C_{\gamma_{s_0}}) \not\subset H$  by construction. An application of (ii) yields a contradiction. Thus,  $S = \emptyset$ .

Let  $\rho_0 = \text{arccot } \kappa_0$  and  $r = \frac{\pi}{2} - \rho_0$ . Choosing  $\delta > 0$  so that  $\text{Im}(\gamma_\delta) \subset B_d(h_\gamma; r)$ , and proceeding as in the proof of (6.1), we can extend  $s \mapsto \gamma_s$  ( $s \in [\delta, 1]$ ) to all of  $[0, 1]$  so that  $\gamma_0$  is a parametrized circle and  $\text{Im}(\gamma_s) \subset B_d(h_\gamma; r)$  for all  $s \in [0, \delta]$  (where  $d$  denotes the distance function on  $\mathbf{S}^2$ ). The inequality  $d(\eta(t), C_\eta(t, \theta)) = \theta < \rho_0$ , which holds for any  $\eta \in \mathcal{L}_{\kappa_0}^{+\infty}$ , implies that

$$d(h_\gamma, C_{\gamma_s}(t, \theta)) < \frac{\pi}{2} \quad \text{for any } t \in [0, 1], \theta \in [0, \rho_0] \text{ and } s \in [0, \delta].$$

Hence  $\text{Im}(C_{\gamma_s}) \subset H$  for all  $s \in [0, \delta]$ . The same inclusion for  $s \in [\delta, 1]$  was established above, so the proof is complete.  $\square$

**(6.6) Corollary.** *Let  $\kappa_0 > 0$  and  $1 \leq \nu \in \mathbf{N}$ .*

- (a) *The subset  $\mathcal{O}$  (resp.  $\mathcal{O}_\nu$ ) of  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$  consisting of all condensed curves (resp. all condensed curves having rotation number  $\nu$ ) is the closure of an open set.*

(b) If  $\gamma \in \mathcal{O}_\nu$  and  $\mathcal{U} \subset \mathcal{L}_{\kappa_0}^{+\infty}(I)$  is any open set containing  $\gamma$ , then  $\gamma$  is homotopic to a smooth curve within  $\mathcal{O}_\nu \cap \mathcal{U}$ .

*Proof.* Let  $\mathcal{S} \subset \mathcal{O}$  be the subset consisting of all curves  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  such that  $\text{Im}(C_\gamma)$  is contained in an open hemisphere. Then  $\mathcal{S}$  is open, because if the compact set  $C = \text{Im}(C_\gamma)$  is such that  $\langle c, h \rangle > 0$  for some  $h \in \mathbf{S}^2$  and all  $c \in C$ , then the same inequality holds for all  $c \in \text{Im}(C_\eta)$  whenever  $\eta \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  is sufficiently close to  $\gamma$ . Similarly,  $\mathcal{O}$  is closed. For if  $\gamma \notin \mathcal{O}$ , then, by (11.2) and (11.5), we can find a 3-dimensional simplex with vertices in  $\text{Im}(C_\gamma)$  containing  $0 \in \mathbf{R}^3$  in its interior. If  $\eta \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  is sufficiently close to  $\gamma$  then we can also find a simplex  $\Delta_\eta$  with vertices in  $\text{Im}(C_\eta)$  such that  $0 \in \text{Int } \Delta_\eta$ . It follows that  $\bar{\mathcal{S}} \subset \mathcal{O}$ .

Let  $\gamma \in \mathcal{O}$ . Define a family  $T^s: \mathbf{S}^2 \rightarrow \mathbf{S}^2$  of Möbius transformations by (3), where  $\text{pr}: \mathbf{S}^2 \rightarrow \mathbf{R}^2$  denotes stereographic projection from  $-h_\gamma$ , and  $h_\gamma$  is the barycenter of the set of closed hemispheres which contain  $C = \text{Im}(C_\gamma)$  (cf. (4.10)). Then  $\gamma_s = T^s \circ \gamma \in \mathcal{S}$  for all  $s \in (0, 1)$  by (6.2), establishing the reverse inclusion  $\bar{\mathcal{S}} \supset \mathcal{O}$ . The proof of the assertion about  $\mathcal{O}_\nu$  is analogous and will be omitted.

To prove (b), let  $\varepsilon > 0$  be such that  $\gamma_s = T^s \circ \gamma \in \mathcal{U}$  for all  $s \in [1 - \varepsilon, 1]$ . Choose a path-connected neighborhood  $\mathcal{V} \subset \mathcal{S} \cap \mathcal{U}$  of  $\gamma_{1-\varepsilon}$ , and, for  $s \in [0, 1 - \varepsilon]$ , let  $\gamma_s$  be a path in  $\mathcal{V}$  joining a smooth curve  $\gamma_0$  to  $\gamma_{1-\varepsilon}$ . As each  $\gamma_s$  is condensed ( $s \in [0, 1]$ ),  $\nu(\gamma_s)$  is defined for all  $s$ ; since it can only take on integral values, it must be independent of  $s$ . Thus,  $s \mapsto \gamma_s$  ( $s \in [0, 1]$ ) is the desired path.  $\square$

### Condensed curves in $\mathcal{L}_{\kappa_0}^{+\infty}$ for $\kappa_0 < 0$

The purpose of this subsection is to prove the following result.

**(6.7) Proposition.** *Let  $\kappa_0 < 0$  and  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  be a condensed curve. Then  $\gamma$  lies in the same connected component of  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$  as a circle traversed a number of times.*

Let  $1 \leq \nu \in \mathbf{N}$  and let  $\mathbf{S}_\nu^2$  denote the  $\nu$ -sheeted connected covering of  $\mathbf{S}^2 \setminus \{\pm \text{point}\}$ , where we may assume that the point is the north pole  $N$ . We will identify  $\mathbf{S}^1 \times (-\frac{\pi}{2}, \frac{\pi}{2})$  with  $\mathbf{S}^2 \setminus \{\pm N\}$  through the homeomorphism  $h$  given by  $h(z, \phi) = (\cos \phi z, \sin \phi)$ . This, in turn, yields an identification of  $\mathbf{S}_\nu^2$  with  $\mathbf{S}_\nu^1 \times (-\frac{\pi}{2}, \frac{\pi}{2})$ , where  $\mathbf{S}_\nu^1$  is the  $\nu$ -sheeted connected covering space of  $\mathbf{S}^1$ . We will prefer to work with the space  $\mathbf{S}_\nu^1 \times (-\frac{\pi}{2}, \frac{\pi}{2})$  instead of  $\mathbf{S}_\nu^2$ , but its Riemannian metric is the one induced on the latter space by  $\mathbf{S}^2$  through the covering map.

**(6.8) Definition.**<sup>3</sup> Let  $0 < R < \frac{\pi}{2}$ . An *acceptable band*  $A: [0, 1] \times [0, 1] \rightarrow \mathbf{S}_\nu^1 \times (-\frac{\pi}{2}, \frac{\pi}{2}) \equiv \mathbf{S}_\nu^2$  is a map given by

$$A(t, u) = (\exp(2\pi\nu it), (1-u)\theta_-(t) + u\theta_+(t)) \quad (t, u \in [0, 1]) \quad (4)$$

and satisfying the following conditions:

- (i)  $\theta_\pm: [0, 1] \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$  are continuous,  $0 \leq \theta_+ \leq R$  and  $-R \leq \theta_- \leq 0$ .
- (ii) Let  $\partial A_+$  (resp.  $\partial A_-$ ) denote the image of  $[0, 1] \times \{1\}$  (resp.  $[0, 1] \times \{0\}$ ) under  $A$ . Then  $d(p, \partial A_-) \geq R$  and  $d(q, \partial A_+) \geq R$  for every  $p \in \partial A_+$  and every  $q \in \partial A_-$ .<sup>4</sup>

The *interior*  $\overset{\circ}{A}$  of  $A$  is simply the interior of the image of  $A$ . The set of all acceptable bands (for fixed  $R$ ) will be denoted by  $\mathcal{A}$  and furnished with the  $C^0$  (uniform) topology. Finally, we denote by  $\mathcal{G}$  the subspace of  $\mathcal{A}$  consisting of all acceptable bands  $A$  such that  $d(p, \partial A_-) = R = d(q, \partial A_+)$  for any  $p \in \partial A_+$  and  $q \in \partial A_-$ . Such a band will be called *good* and  $R$  its *width*.

The motivation for this definition comes from the following lemma.

**(6.9) Lemma.** Let  $\kappa_0 = \cot \rho_0 < 0$  and  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  be a condensed curve having rotation number  $\nu$ . Then the image of the lift of the regular band  $B_\gamma: [0, 1] \times [\rho_0 - \pi, 0] \rightarrow \mathbf{S}^2$  of  $\gamma$  to  $\mathbf{S}_\nu^2$  is the image of a good band of width  $\pi - \rho_0$ .

*Proof.* By hypothesis, the image of the caustic band  $C_\gamma$  is contained in a hemisphere, say,

$$H = \{p \in \mathbf{S}^2 : \langle p, N \rangle \geq 0\}.$$

Let  $\hat{\gamma}$  be the other boundary curve of  $B_\gamma$ ,  $\hat{\gamma}(t) = B_\gamma(t, \rho_0 - \pi)$ . Then  $\hat{\gamma}(t) = -C_\gamma(t, \rho_0) \in -H$  for all  $t \in [0, 1]$ . Since  $d(\gamma(t), \hat{\gamma}(t)) = \pi - \rho_0 < \frac{\pi}{2}$ ,  $\text{Im}(\gamma) \subset H$  and  $\text{Im}(\hat{\gamma}) \subset -H$ , the image of the regular band is actually contained in  $\mathbf{S}^1 \times [\rho_0 - \pi, \pi - \rho_0]$  (where we are identifying  $\mathbf{S}^2 \setminus \{\pm N\}$  with  $\mathbf{S}^1 \times (-\frac{\pi}{2}, \frac{\pi}{2})$ ).

Let  $\tilde{B}_\gamma: [0, 1] \times [\rho_0 - \pi, 0] \rightarrow \mathbf{S}_\nu^2$  be the lift of  $B_\gamma$  to  $\mathbf{S}_\nu^2 \equiv \mathbf{S}_\nu^1 \times (-\frac{\pi}{2}, \frac{\pi}{2})$ . For each  $z \in \mathbf{S}_\nu^1$ , let the *meridian*  $\mu_z$  be the geodesic parametrized by  $\mu_z(t) = (z, t)$ ,  $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ . By what we have just proved and the fact that  $\gamma$  has rotation number  $\nu$ , we may define continuous functions  $\theta_\pm: \mathbf{S}_\nu^1 \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$  by the relations

$$\mu_z(\theta_+(z)) \in \tilde{B}_\gamma([0, 1] \times \{0\}) \quad \text{and} \quad \mu_z(\theta_-(z)) \in \tilde{B}_\gamma([0, 1] \times \{\rho_0 - \pi\}).$$

<sup>3</sup>These notions will only be used in this subsection.

<sup>4</sup>Here and in what follows,  $d$  denotes the distance function on  $\mathbf{S}_\nu^2$  (or on  $\mathbf{S}^2$ ).



Then the map  $A: [0, 1] \times [0, 1] \rightarrow \mathbf{S}_\nu^1 \times (-\frac{\pi}{2}, \frac{\pi}{2}) \equiv \mathbf{S}_\nu^2$  given by

$$A(t, u) = (\exp(2\pi\nu it), (1-u)\theta_-(t) + u\theta_+(t)) \quad (t, u \in [0, 1])$$

defines an acceptable band whose image coincides with that of  $\tilde{B}_\gamma$ . Furthermore, the equality  $d(\gamma(t), \hat{\gamma}(t)) = \pi - \rho_0$  implies that  $d(p, \partial A_\pm) \leq \pi - \rho_0$  for any  $p \in \partial A_\mp$ . We claim that  $A$  is a good band of width  $\pi - \rho_0$ . To see this, suppose  $\eta: [0, 1] \rightarrow \mathbf{S}_\nu^2$  is a piecewise  $C^1$  curve joining  $\partial A_-$  to  $\partial A_+$  and write  $\eta(u) = \tilde{B}_\gamma(t(u), \theta(u))$ . Then the length is minimized when  $\theta$  is monotone and  $\dot{t}(u) = 0$  for all  $u \in [0, 1]$ , hence the minimal length is  $\pi - \rho_0$ ; we omit the details since an entirely similar argument is presented in the proof of (10.5).  $\square$

**(6.10) Lemma.** *The space  $\mathcal{A}$  is contractible.*

*Proof.* Let  $A \in \mathcal{A}$  be given by (4) and let  $s \in [0, 1]$ . Define a family of acceptable bands  $A_s$  by

$$A_s(t, u) = (\exp(2\pi\nu it), (1-u)\theta_-^s(t) + u\theta_+^s(t)),$$

where

$$\theta_+^s(t) = (1-s)\theta_+(t) + sR \quad \text{and} \quad \theta_-^s(t) = (1-s)\theta_-(t) - sR$$

Then the map  $\mathcal{A} \times [0, 1] \rightarrow \mathcal{A}$  given by  $(A, s) \mapsto A_s$  is a contraction of  $\mathcal{A}$ .  $\square$

**(6.11) Lemma.** *The subspace  $\mathcal{G}$  is a retract of  $\mathcal{A}$ .*

*Proof.* Let  $A \in \mathcal{A}$  be given by (4). Define  $A^1 = \text{Im}(A)$ ,  $\theta_\pm^1 = \theta_\pm$  and

$$A^2 = \{p \in A^1 : d(p, \partial A_-^1) \leq R + \frac{1}{2}\}.$$

We will call a geodesic  $\mu_z$  in  $\mathbf{S}_\nu^2 \equiv \mathbf{S}_\nu^1 \times (-\frac{\pi}{2}, \frac{\pi}{2})$  of the form  $\{z\} \times (-\frac{\pi}{2}, \frac{\pi}{2})$  a *meridian*, and parametrize it by  $\mu_z(t) = (z, t)$ . We begin by establishing the following facts:

- (a) Each meridian  $\mu_z$  intersects  $\partial A^2$  at exactly two points  $\mu_z(\theta_-^2(z))$  and  $\mu_z(\theta_+^2(z))$ , with  $\theta_+^2 \geq 0$  and  $\theta_-^2 \leq 0$ . We define  $\partial A_\pm^2$  as the set of all  $\mu_z(\theta_\pm^2(z))$  for  $z \in \mathbf{S}_\nu^1$ .
- (b)  $\partial A_-^2 = \partial A_-^1$ .
- (c)  $p \in \partial A_+^2$  if and only if one of the following holds:

$$\begin{aligned} p \in \partial A_+^1 \quad \text{and} \quad d(p, \partial A_-^1) \leq R + \frac{1}{2}, \quad \text{or} \\ p \in \mathring{A}^1 \quad \text{and} \quad d(p, \partial A_-^1) = R + \frac{1}{2}. \end{aligned}$$

(d) The boundary  $\partial A^2$  of  $A^2$  is the disjoint union of  $\partial A_+^2$  and  $\partial A_-^2$ . Moreover,

$$R \leq d(p, \partial A_-^2) \leq R + \frac{1}{2} \quad \text{and} \quad R \leq d(q, \partial A_+^2) \leq d(q, \partial A_+^1)$$

for any  $p \in \partial A_+^2$  and  $q \in \partial A_-^2$ .

(e)  $A^2$  is the (image of) an acceptable band, and the functions in (6.8(i)) corresponding to  $A^2$  are  $\theta_\pm^2$ . Moreover,

$$0 \leq \theta_+^2 \leq \min\{R + \frac{1}{2}, \theta_+^1\} \quad \text{and} \quad -R \leq \theta_-^2 = \theta_-^1 \leq 0. \quad (5)$$

The inclusion  $\partial A_-^1 \subset \mathbf{S}_\nu^1 \times [-R, 0]$  implies, firstly, that

$$A^2 \cap (\mathbf{S}_\nu^1 \times [-R, 0]) = A^1 \cap (\mathbf{S}_\nu^1 \times [-R, 0]), \quad (6)$$

as every point of  $A^1 \cap (\mathbf{S}_\nu^1 \times [-R, 0])$  lies at a distance less than or equal to  $R$  from  $\partial A_-^1$ . Secondly, it implies that

$$t \mapsto d(\mu_z(t), \partial A_-^1)$$

is a monotone decreasing function of  $t$  when  $t \geq 0$ .

It follows from (6) and the properties of  $A^1$  that, for any  $z \in \mathbf{S}_\nu^1$ , there exists a unique  $\theta_-^2(z) \in [-R, 0]$  such that  $\mu_z(\theta_-^2(z)) \in \partial A^2$ , unless  $\mu_z(0) \in \partial A_+^1$ . In the latter case,  $d(\mu_z(0), \partial A_-^1) = R$ ,  $\theta_-^2(z) = -R$  and  $\theta_+^2(z) = 0$ . If  $\mu_z(0) \notin \partial A_+^1$ , let  $\theta_+^2(z) > 0$  be the smallest  $t \in (0, R]$  such that either  $\mu_z(t) \in \partial A_+^1$  or  $d(\mu_z(t), \partial A_-^1) = R + \frac{1}{2}$ . Suppose  $\mu_z(\theta_+^2(z)) \in \partial A_+^1$ . Then  $\mu_z(\theta_+^2(z)) \in A^2$  (because it lies a distance  $\leq R + \frac{1}{2}$  from  $\partial A_-^1$ ), while  $\mu_z(t) \notin A^1 \supset A^2$  for  $t > \theta_+^2(z)$ . Thus,  $\mu_z(\theta_+^2(z)) \in \partial A^2$ . If  $d(\mu_z(\theta_+^2(z)), \partial A_-^1) = R + \frac{1}{2}$ , then again  $\mu_z(\theta_+^2(z)) \in A^2$  while  $\mu_z(t) \notin A^2$  for  $t > \theta_+^2(z)$ , since, for such  $t$ ,  $d(\mu_z(t), \partial A_-^1) > R + \frac{1}{2}$  by the second consequence. Moreover, in both cases  $\mu_z(t)$  does not intersect  $\partial A^2$  again for  $t > 0$ . This proves (a), (b), (c) and also establishes (5).

Since

$$\partial A^2 = \bigcup_{z \in \mathbf{S}_\nu^1} \mu_z \cap \partial A^2,$$

(a) implies the first assertion of (d). In turn, (b) and (c) together immediately imply that

$$R \leq d(p, \partial A_-^2) = d(p, \partial A_-^1) \leq R + \frac{1}{2}$$

for any  $p \in \partial A_+^2$ . That  $d(q, \partial A_+^2) \leq d(q, \partial A_+^1)$  for any  $q \in \partial A_-^2$  follows from the fact that  $\partial A_+^2$  lies below  $\partial A_+^1$ , in the sense that any geodesic joining  $\partial A_-^1$  to  $\partial A_+^1$  must first intersect a point of  $\partial A_+^2$ . Indeed,  $\theta_+^2(z) \leq \theta_+^1(z)$  for any

$z \in \mathbf{S}_\nu^1$ , as we have already seen in (5). Thus, (d) holds.

By construction,

$$A^2 = \{p \in \mathbf{S}_\nu^2 \equiv \mathbf{S}_\nu^1 \times (-\frac{\pi}{2}, \frac{\pi}{2}) : p = (z, \theta) \text{ for some } \theta \in [\theta_-^2(z), \theta_+^2(z)]\}.$$

Hence,  $A^2$  is the image of the acceptable band given by

$$(t, u) \mapsto (\exp(2\pi\nu it), (1-u)\theta_-^2(t) + u\theta_+^2(t)) \quad (t, u \in [0, 1]).$$

Using induction and the corresponding versions of items (a)–(e) (whose proofs are the same in the general case), define

$$A^{n+1} = \{p \in A^n : d(p, \partial A_{(-1)^n}^n) \leq R + 2^{-n}\} \quad (n \in \mathbf{N}).$$

Finally, let  $B = \bigcap_{n=1}^{+\infty} A^n$ . We claim that  $B$  is the image of a good band.

Given  $N \in \mathbf{N}$  and  $m, n > N$ , we have

$$|\theta_\pm^n(z) - \theta_\pm^m(z)| \leq 2^{-N+1} \quad \text{for any } z \in \mathbf{S}_\nu^1$$

by construction. Therefore,  $\theta_+^n \searrow \theta_+$  and  $\theta_-^n \nearrow \theta_-$  for some functions  $\theta_\pm : \mathbf{S}_\nu^1 \rightarrow [-R, R]$ , which are continuous as the uniform limit of continuous functions. Moreover,  $B$  is the image of the map

$$(t, u) \mapsto (\exp(2\pi\nu it), (1-u)\theta_-(t) + u\theta_+(t)) \quad (t, u \in [0, 1]),$$

again by construction. We claim that  $d(x, \partial B_\pm) = R$  for any  $x \in \partial B_\mp$ . Suppose for a contradiction that  $d(p, \partial B_-) < R$  for some  $p \in \partial B_+$ , and let  $pq$  be a geodesic of length  $d(p, \partial B_-)$ , with  $q \in \partial B_-$ . Choose neighborhoods  $U \ni p$  and  $V \ni q$  such that  $d(x, y) < R$  for any  $x \in U$ ,  $y \in V$ . Since  $p, q \in \partial B_\pm$ , by choosing a sufficiently large  $n \in \mathbf{N}$ , we may find  $x \in \partial A_+^n \cap U$  and  $y \in \partial A_-^n \cap V$  with  $d(x, y) < R$ , a contradiction. Similarly, if  $d(p, \partial B_-) = R + \varepsilon$  for some  $\varepsilon > 0$ , choose neighborhoods  $U \ni p$  and  $V \ni q$  such that  $d(x, y) \geq R + \frac{\varepsilon}{2}$  for any  $x \in U$  and  $y \in V$ . Let  $N \in \mathbf{N}$  be so large that  $2^{-N} < \frac{\varepsilon}{2}$ . Since  $p, q \in \partial B_\pm$ , we may find some  $n > 2N$  and  $x \in \partial A_+^n \cap U$ ,  $y \in \partial A_-^n \cap V$ . Then  $d(x, y) \geq R + \frac{\varepsilon}{2} > R + 2^{-N}$ , again a contradiction. The assumption that  $d(q, \partial B_+) \neq R$  for some  $q \in \partial B_-$  also yields a contradiction. We conclude that  $B$  is a good band of width  $R$ .

If  $r : \mathcal{A} \rightarrow \mathcal{G}$  is the map which associates to an acceptable band  $A$  the good band  $B$  obtained by the process described above, then  $r(A) = A$  whenever  $A \in \mathcal{G}$ . In addition, we see by induction that the map  $A \mapsto A^n$  is continuous on  $\mathcal{A}$  for every  $n \in \mathbf{N}$ . Given  $\varepsilon > 0$ , we can arrange that  $\|A^n - A^m\|_{C^0} < \varepsilon$  for any  $A \in \mathcal{A}$  by choosing  $m, n \geq N$  and a sufficiently large  $N \in \mathbf{N}$ . Hence,

$r: \mathcal{A} \rightarrow \mathcal{G}$  is a retraction. □

**(6.12) Corollary.** *The space  $\mathcal{G}$  is contractible.*

*Proof.* This is an immediate consequence of (6.10) and (6.11). □

**(6.13) Definition.** Let  $B$  be a good band of width  $R$ . A *track* of  $B$  is a curve on  $\mathbf{S}^2_\nu$  of length  $R$  joining a point of  $\partial B_+$  to a point of  $\partial B_-$ .

In other words, a track is a length-minimizing geodesic joining  $\partial B_+$  to  $\partial B_-$ ; in particular, it is a smooth curve. Also, if  $\Gamma_1, \Gamma_2$  are tracks through  $p \in \partial B_+$  and  $q \in \partial B_-$  then  $\Gamma_1 = \Gamma_2$ , since two geodesics on  $\mathbf{S}^2$  intersect at a pair of antipodal points, and  $p$  and  $q$  do not map to the same point nor to a pair of antipodal points on  $\mathbf{S}^2$  under the covering map.

**(6.14) Lemma.** *Let  $B$  be a good band. Then two tracks of  $B$  cannot intersect at a point lying in  $\mathring{B}$ .*

*Proof.* Suppose for the sake of obtaining a contradiction that two tracks  $p_1q_1$  and  $p_2q_2$ , with  $p_i \in \partial B_+$  and  $q_i \in \partial B_-$ , intersect at a point  $x \in \mathring{B}$  (see fig. 10). Then one of the following must occur:<sup>5</sup>

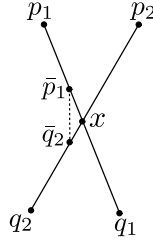


Figure 10:

- (i)  $xq_1 = xq_2$ ;
- (ii)  $xq_1 > xq_2$ ;
- (iii)  $xq_1 < xq_2$ .

If (i) holds, let  $\bar{p}_1, \bar{q}_2$  be points on  $p_1x$  and  $xq_2$ , respectively, which lie in a normal neighborhood of  $x$ . Then, by the triangle inequality,

$$R = p_1q_1 = p_1x + xq_2 > p_1\bar{p}_1 + \bar{p}_1\bar{q}_2 + \bar{q}_2q_2.$$

This contradicts the fact that  $B$  is a good band of width  $R$ .

<sup>5</sup>Here  $ab$  denotes the segment of the corresponding geodesic and also its length.

If (ii) holds then  $R = p_1q_1 > p_1x + xq_2$ . Again, this contradicts the fact that  $p_1q_1$  is a path of minimal length joining  $p_1$  to  $\partial B_-$ . Similarly, if (iii) holds then  $R = p_2q_2 > p_2x + xq_1$ , contradicting the fact that  $p_2q_2$  is a path of minimal length joining  $p_2$  to  $\partial B_-$ .  $\square$

*Remark.* Note that this result may be false for an acceptable band. In the proof, we have implicitly used the fact that if  $pq$  is a path of minimal length joining  $p \in \partial B_+$  to  $\partial B_-$  then  $pq$  is also a path of minimal length joining  $q$  to  $\partial B_+$ , and this is not necessarily true for an acceptable band.

**(6.15) Lemma.** *Every point in the interior of a good band  $B$  lies in a unique track of  $B$ .*

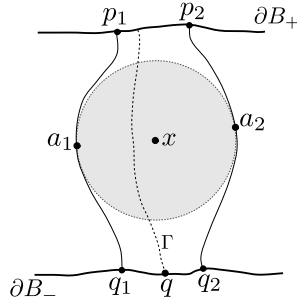


Figure 11:

*Proof.* Let  $R$  be the width of  $B$  and let  $T \subset \text{Im}(B)$  consist of all points which lie on some track of  $B$ . It is clear from the definitions that  $\partial B_{\pm} \subset T$ . We claim that  $a \in T$  if and only if

$$d(a, \partial B_+) + d(a, \partial B_-) = R \quad (7)$$

The existence of a track through  $a$  implies that  $d(a, \partial B_+) + d(a, \partial B_-) \leq R$ . If the inequality were strict, then there would exist a path of length less than  $R$  joining  $\partial B_+$  to  $\partial B_-$ , which is impossible. Conversely, suppose (7) holds, and let  $p \in \partial B_+$ ,  $q \in \partial B_-$  be the points of  $\partial B_+$  (resp.  $\partial B_-$ ) which are closest to  $a$ . Then the concatenation of the geodesics  $pa$  and  $aq$  is a path of length  $R$  joining  $\partial B_+$  to  $\partial B_-$ , i.e., a track. Hence,  $a \in T$ .

The characterization of  $T$  that we have established implies that the latter is a closed set. Now suppose that  $x \notin T$ , let  $V$  be the component of  $\mathring{B} \setminus T$  containing  $x$  (see fig. 11, where  $V$  is depicted as a gray open ball). Since  $T$  is closed, any point in  $\partial V$  lies in  $T$ . Choose points  $a_1, a_2 \in \partial V \setminus (\partial B_+ \cup \partial B_-)$  such that the (unique) tracks  $p_iq_i$  going through  $a_i$  do not coincide, where  $p_i \in \partial B_+$  and  $q_i \in \partial B_-$  ( $i = 1, 2$ ). Such points  $a_i$  exist because otherwise  $V = \mathring{B}$ , which

is absurd since any point on a track lies in  $T$ . Because the tracks are distinct, at least one of  $p_1 \neq p_2$  or  $q_1 \neq q_2$  must hold. Assume without loss of generality that  $q_1 \neq q_2$ , and let  $q \in \partial B_-$  be such that it is possible to join  $q$  to  $x$  in  $\text{Im}(B)$  without crossing  $p_1q_1$  nor  $p_2q_2$ . Let  $\Gamma$  be a track through  $q$ . Then  $\Gamma$  joins  $q$  to  $\partial B_+$ , but it does not intersect  $p_1q_1$  nor  $p_2q_2$  by (6.14). It follows that  $\Gamma$  must contain points of  $V$ , a contradiction which shows that  $T = \text{Im}(B)$ . In other words, every point of  $\text{Im}(B)$  lies in a track of  $B$ ; uniqueness has already been established in (6.14).  $\square$

**(6.16) Corollary.** *Let  $B$  be a good band of width  $R$ . Then  $d(a, \partial B_+) + d(a, \partial B_-) = R$  for any  $a \in \text{Im}(B)$ .*  $\square$

**(6.17) Lemma.** *Let  $B$  be a good band of width  $R$  and let  $0 < r < R$ . Then the set  $\gamma_r$  consisting of all those points in  $\mathring{B}$  at distance  $r$  from  $\partial B_+$  is (the image of) a closed admissible curve whose radius of curvature  $\rho$  satisfies  $r \leq \rho \leq \pi - R + r$  almost everywhere.*

*Proof.* For  $p \in \mathring{B}$ , let  $\Gamma_p: [0, R] \rightarrow \mathbf{S}_v^2$  denote the unique track through  $p$ , parametrized by arc-length, with  $\Gamma_p(0) \in \partial B_-$  and  $\Gamma_p(R) \in \partial B_+$ . Define vector fields  $\mathbf{n}$  and  $\mathbf{t}$  on  $\mathring{B}$  by letting  $\mathbf{n}(p)$  be the unit tangent vector to  $\Gamma_p$  at  $p$  and  $\mathbf{t}(p) = \mathbf{n}(p) \times p$ . We claim that the restriction of  $\mathbf{n}$  (and consequently that of  $\mathbf{t}$ ) to any compact subset  $K$  of  $\mathring{B}$  satisfies a Lipschitz condition. Let  $d_0 < \min\{d(K, \partial B_+), d(K, \partial B_-)\}$ , let  $a_0, a_1 \in K$ , with  $a_1$  close to  $a_0$ , and consider the (spherical) triangle having  $\Gamma_{a_0}, \Gamma_{a_1}, a_0a_1$  as sides and  $a_0, a_1, a_2$  as vertices (see fig. 12). The point  $a_2$  must lie outside of  $\mathring{B}$  by (6.14). Let  $p_0$  be the point where the geodesic segment  $a_0a_2$  intersects  $\partial B_\pm$ . Then

$$a_0a_2 \geq a_0p_0 \geq d_0.$$

Hence, by the law of sines (for spherical triangles) applied to  $\triangle a_0a_1a_2$ ,

$$\frac{\sin a_2}{\sin(a_0a_1)} = \frac{\sin a_1}{\sin(a_0a_2)} \leq \frac{1}{\sin d_0},$$

Using parallel transport we may compare

$$\frac{\angle(\mathbf{n}(a_0), \mathbf{n}(a_1))}{a_0a_1} \quad \text{with} \quad \frac{\angle a_2}{a_0a_1} \approx \frac{\sin a_2}{\sin(a_0a_1)}$$

to obtain a Lipschitz condition satisfied by the former, but we omit the computations.

Now given  $p \in \mathring{B}$  at distance  $r$  from  $\partial B_+$ ,  $0 < r < R$ , let  $\gamma_r$  be the integral curve through  $p$  of the vector field  $\mathbf{t}$ . Then  $\gamma_r$  is parametrized by arc-length

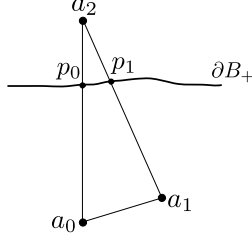


Figure 12:

and its frame is given by

$$\Phi_{\gamma_r}(t) = \begin{pmatrix} | & | & | \\ \gamma_r(t) & \mathbf{t}(\gamma_r(t)) & \mathbf{n}(\gamma_r(t)) \\ | & | & | \end{pmatrix}$$

by construction. If  $d(t) = d(\gamma_r(t), \partial B_+)$  then  $\dot{d} \equiv 0$ , since  $\mathbf{t}(\gamma_r(t))$  is orthogonal to the track through  $\gamma_r(t)$  for every  $t$ . Hence  $d$  is constant, equal to  $r$ , and  $\gamma_r$  is a closed curve. Moreover, since  $\mathbf{t}$  and  $\mathbf{n}$  satisfy a Lipschitz condition when restricted to the image of  $\gamma_r$ , we see that the entries of  $\Phi_{\gamma_r}$  are absolutely continuous with bounded derivative. In particular, these derivatives belong to  $L^2$ . We conclude that  $\gamma_r$  is admissible.

For  $r - R < \theta < r$ , the curve  $\gamma_{r-\theta}$  is the translation of  $\gamma_r$  by  $\theta$  (as defined on p. 24, eq. (8)) by construction. Since  $\gamma_{r-\theta}$  is also regular, we deduce from (6) in (4.7) that the radius of curvature  $\rho$  of  $\gamma_r$  satisfies

$$0 < \rho(t) - \theta < \pi$$

for all  $t$  at which  $\rho$  is defined and all  $\theta$  in  $(r - R, r)$ . Therefore,  $r \leq \rho \leq \pi - R + r$  a.e..  $\square$

**(6.18) Corollary.** *Let  $B$  be a good band of width  $R$  and let  $0 < r < R$ . Then the central curve  $\gamma_{\frac{R}{2}}$  is an admissible curve whose radius of curvature is restricted to  $[\frac{R}{2}, \pi - \frac{R}{2}]$ .*  $\square$

Before finally presenting a proof of (6.7), we extend the definition of the regular band of a curve to any space  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ .

**(6.19) Definition.** Let  $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}$ . The (regular) band  $B_\gamma$  spanned by  $\gamma$  is the map:

$$B_\gamma: [0, 1] \times [\rho_1 - \pi, \rho_2] \rightarrow \mathbf{S}^2, \quad B_\gamma(t, \theta) = \cos \theta \gamma(t) + \sin \theta \mathbf{n}(t).$$

The statement and proof of (4.7) still hold, except for obvious modifications.

*Proof of (6.7).* By (2.10), we may assume that  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  ( $\kappa_0 < 0$ ) is of class  $C^2$ . Let  $\rho_\gamma$  denote its radius of curvature,  $\rho_0 = \operatorname{arccot} \kappa_0$ ,

$$\rho_1 = \frac{\pi - \rho_0}{2}, \quad \kappa_1 = \cot \rho_1 \quad (8)$$

(compare (2.26)) and let  $\eta$  be the translation of  $\gamma$  by  $\rho_1$ . Then the radius of curvature  $\rho_\eta$  of  $\eta$  satisfies  $\rho_1 < \rho_\eta < \pi - \rho_1$ . Since  $\rho_\eta$  is continuous, there exists  $\bar{\rho}_1$  with  $\rho_1 < \bar{\rho}_1 < \frac{\pi}{2}$  such that

$$\bar{\rho}_1 < \rho_\eta < \pi - \bar{\rho}_1.$$

In particular, the regular band of  $\eta$  may be extended from  $[0, 1] \times [-\rho_1, \rho_1]$  to  $[0, 1] \times [-\bar{\rho}_1, \bar{\rho}_1]$ . Consider the space  $\mathcal{G}$  of good bands of width  $R = 2\bar{\rho}_1$  and the corresponding space  $\mathcal{A} \supset \mathcal{G}$  of acceptable bands. Let  $B_0$  the regular band of  $\eta$  (whose image is the same as that of the regular band of  $\gamma$ ), and  $B_1$  be the regular band of a condensed circle in  $\mathcal{L}_{\kappa_0}^{+\infty}$  traversed  $\nu$  times, where  $\nu$  is the rotation number of  $\gamma$ . The combination of (6.9), (6.12) and (6.18) yields a homotopy  $s \mapsto \eta_s$  from  $\eta = \eta_0$  to a circle  $\eta_1$  traversed  $\nu$  times, where  $\eta_s$  is the central curve of a good band  $B_s$ ,  $s \in [0, 1]$ . Moreover, (6.18) guarantees that the radius of curvature  $\rho_{\eta_s}$  of  $\eta_s$  satisfies  $\bar{\rho}_1 \leq \rho_{\eta_s} \leq \pi - \bar{\rho}_1$  for each  $s \in [0, 1]$ . Consequently,

$$\rho_1 < \rho_{\eta_s} < \pi - \rho_1 \quad \text{for each } s \in [0, 1]$$

and it follows that  $s \mapsto \eta_s$  is a path in  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}$  from  $\eta$  to a parametrized circle. If we let  $\gamma_s$  be the translation of  $\eta_s$  by  $-\rho_1$ , then  $\gamma_0$  is the original curve  $\gamma$ , and  $s \mapsto \gamma_s$  is a path in  $\mathcal{L}_{\kappa_0}^{+\infty}$  from  $\gamma$  to a circle  $\gamma_1$  traversed  $\nu$  times.

We have proved that  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  lies in the same component of  $\mathcal{L}_{\kappa_0}^{+\infty}$  as a circle traversed a number of times. The latter space may be replaced by  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$  without altering the conclusion by the usual trick of substituting  $\gamma_s$  by  $\Phi_{\gamma_s}(0)^{-1}\gamma_s$  ( $s \in [0, 1]$ ).  $\square$



## 7

### Non-diffuse Curves

In this section we define a notion of rotation number for any non-diffuse curve in  $\mathcal{L}_{\kappa_0}^{+\infty}$  and prove a bound on the total curvature of such a curve which depends only on its rotation number and  $\kappa_0$  (prop. (7.8)).

**(7.1) Lemma.** *Suppose  $X$  is a connected, locally connected topological space and  $C \neq \emptyset$  is a closed connected subspace. Let  $\sqcup_{\alpha \in J} B_\alpha$  be the decomposition of  $X \setminus C$  into connected components. Then:*

(a)  $\partial B_\alpha \subset C$  for all  $\alpha \in J$ .

(b) For any  $J_0 \subset J$ , the union  $C \cup \bigcup_{\beta \in J_0} B_\beta$  is also connected.

*Proof.* Assume (a) is false, and let  $p \in \partial B_\alpha \setminus C$  for some  $\alpha$ . Since  $C$  is closed and  $X$  locally connected, we can find a connected neighborhood  $U \ni p$  which is disjoint from  $C$ . But  $U \cap B_\alpha \neq \emptyset$  and  $B_\alpha$  is a connected component of  $X \setminus C$ , hence  $U \subset B_\alpha$ , contradicting the fact that  $p \in \partial B_\alpha$ . Therefore  $\partial B_\alpha \subset C$  as claimed. Moreover,  $\partial B_\alpha \neq \emptyset$ , otherwise  $X = B_\alpha \sqcup (X \setminus B_\alpha)$  would be a decomposition of the connected space  $X$  into two open sets. Now, for  $\beta \in J$ , set  $A_\beta = C \cup B_\beta = C \cup \overline{B}_\beta$ . Each  $A_\beta$  is a union of two connected sets with non-empty intersection, hence is itself connected. Similarly,  $C \cup \bigcup_{\beta \in J} B_\beta = \bigcup_{\beta \in J} A_\beta$  is connected as the union of a family of connected sets with a point in common.  $\square$

We will also need the following well-known results.<sup>1</sup>

**(7.2) Theorem.** *Let  $A \subset \mathbf{S}^2$  be a connected open set.*

(a)  *$A$  is simply-connected if and only if  $\mathbf{S}^2 \setminus A$  is connected.*

(b) *If  $A$  is simply-connected and  $\mathbf{S}^2 \setminus A \neq \emptyset$ , then  $A$  is homeomorphic to an open disk.*

(c) *Let  $S_\pm \subset \mathbf{S}^2$  be disjoint and homeomorphic to  $\mathbf{S}^1$ . Then the closure of the region bounded by  $S_-$  and  $S_+$  is homeomorphic to  $\mathbf{S}^1 \times [-1, 1]$ .  $\square$*

<sup>1</sup>Part (b) of (7.2) is an immediate corollary of the Riemann mapping theorem and part (c) is the 2-dimensional case of the annulus theorem.

**(7.3) Lemma.** *Let  $U_{\pm} \subset \mathbf{S}^2$  be homeomorphic to open disks,  $U_- \cup U_+ = \mathbf{S}^2$ . Then*

$$U_- \cap U_+ \approx \mathbf{S}^1 \times (-1, 1).$$

*Proof.* We first make two claims:

- (a) Suppose  $C \approx \mathbf{S}^1 \times [-1, 1]$  and  $h: \partial C_- \rightarrow \mathbf{S}^1 \times \{-1\}$  is a homeomorphism, where  $\partial C_-$  is one of the boundary circles of  $C$ . Then  $h$  may be extended to a homeomorphism  $H: C \rightarrow \mathbf{S}^1 \times [-1, 1]$ .
- (b) Let  $M$  be a tower of cylinders, in the sense that:
  - (i)  $M_i \approx \mathbf{S}^1 \times [-1, 1]$  for each  $i \in \mathbf{Z}$ ;
  - (ii)  $M = \bigcup_{i \in \mathbf{Z}} M_i$  and  $M$  has the weak topology determined by the  $M_i$ ;
  - (iii)  $M_i \cap M_j = \emptyset$  for  $j \neq i \pm 1$  and  $M_i \cap M_{i+1} = S_i^+ = S_{i+1}^-$ , where  $S_i^{\pm}$  are the boundary circles of  $M_i$ .

Then  $M \approx \mathbf{S}^1 \times (-1, 1)$ .

Claim (a) is obviously true if  $C = \mathbf{S}^1 \times [-1, 1]$ : Just set  $H(z, t) = (h(z), t)$ . In the general case let  $F: C \rightarrow \mathbf{S}^1 \times [-1, 1]$  be a homeomorphism. Note that  $\partial C$  is well-defined as the inverse image of  $\mathbf{S}^1 \times \{\pm 1\}$  ( $p \in \partial C$  if and only if  $U \setminus \{p\}$  is contractible whenever  $U$  is a sufficiently small neighborhood of  $p$ ). Hence  $\partial C$  consists of two topological circles,  $\partial C_{\pm} = F^{-1}(\mathbf{S}^1 \times \{\pm 1\})$ . Let  $f = F|_{\partial C_-}$  and let  $g = h \circ f^{-1}: \mathbf{S}^1 \rightarrow \mathbf{S}^1$ . As we have just seen, we can extend  $g$  to a self-homeomorphism  $G$  of  $\mathbf{S}^1 \times [-1, 1]$ . Now define  $H: C \rightarrow \mathbf{S}^1 \times [-1, 1]$  by  $H = G \circ F$ . Then  $H|_{\partial C_-} = g \circ f = h$ , as desired.

To prove claim (b), let  $H_0: M_0 \rightarrow \mathbf{S}^1 \times [-\frac{1}{2}, \frac{1}{2}]$  be any homeomorphism. By applying (a) to  $M_{\pm 1}$  and  $h_{\pm 1} = H_0|_{S_0^{\pm}}$ , we can extend  $H_0$  to a homeomorphism

$$H_1: M_0 \cup M_{\pm 1} \rightarrow \mathbf{S}^1 \times \left[-\frac{2}{3}, \frac{2}{3}\right],$$

and, inductively, to a homeomorphism

$$H_k: \bigcup_{|i| \leq k} M_i \rightarrow \mathbf{S}^1 \times \left[-1 + \frac{1}{k+2}, 1 - \frac{1}{k+2}\right] \quad (k \in \mathbf{N}).$$

Finally, let  $H: M \rightarrow \mathbf{S}^1 \times (-1, 1)$  be defined by  $H(p) = H_i(p)$  if  $p \in M_i$ . Then  $H$  is bijective, continuous and proper, so it is the desired homeomorphism.

Returning to the statement of the lemma, note first that  $\partial U_{\pm} \subset U_{\mp}$ . Indeed, if  $p \in \partial U_- \cap (\mathbf{S}^2 \setminus U_+)$  then  $p \notin U_- \cup U_+ = \mathbf{S}^2$ , hence no such  $p$  exists. Let  $h_{\pm}: B(0; 1) \rightarrow U_{\pm}$  be homeomorphisms, and define  $f_{\pm}: [0, 1) \rightarrow \mathbf{R}$  by

$$f_{\pm}(r) = \sup \{d(p, \partial U_{\pm}) : p \in h_{\pm}(r\mathbf{S}^1)\},$$

where  $d$  denotes the distance on  $\mathbf{S}^2$ . We claim that  $\lim_{r \rightarrow 1} f_{\pm}(r) = 0$ . Observe first that  $f_{\pm}$  is strictly decreasing, for if  $q \in h_{\pm}(r_0 \mathbf{S}^1)$ ,  $r_0 < r$ , then any geodesic joining  $q$  to  $\partial U_{\pm}$  intersects  $h(r \mathbf{S}^1)$ . Hence the limit exists; if it were positive, then  $U_{\pm}$  would be at a positive distance from  $\partial U_{\pm}$ , which is absurd.

Now choose  $n \in \mathbf{N}$  such that

$$f_{\pm}(t) < \frac{1}{2} \min \{d(\partial U_{-}, \mathbf{S}^2 \setminus U_{+}), d(\partial U_{+}, \mathbf{S}^2 \setminus U_{-})\}$$

for any  $t > 1 - \frac{1}{n}$ . Set

$$S_i = h_{+} \left( \left( 1 - \frac{1}{n+i} \right) \mathbf{S}^1 \right) \text{ for } i > 0 \text{ and } S_i = h_{-} \left( \left( 1 - \frac{1}{n-i} \right) \mathbf{S}^1 \right) \text{ for } i < 0.$$

Finally, let  $M_0$  be the region of  $U_{-} \cap U_{+}$  bounded by  $S_1$  and  $S_{-1}$  and, for  $i > 0$  (resp.  $< 0$ ), let  $M_i$  the region bounded by  $S_i$  and  $S_{i+1}$  (resp.  $S_{i-1}$ ). Using (7.2(c)) we see that  $U_{-} \cap U_{+} = \bigcup M_i$  is a tower of cylinders as in claim (b), and we conclude that  $U_{-} \cap U_{+} \approx \mathbf{S}^1 \times (-1, 1)$ .  $\square$

*Remark.* Another proof of the previous result can be obtained as follows: Since  $U_{\pm}$  are each contractible, the Mayer-Vietoris sequence yields immediately that  $U_{-} \cap U_{+}$  has the homology of  $\mathbf{S}^1$ . Together with a little more work it then follows from the classification of noncompact surfaces that  $U_{-} \cap U_{+} \approx \mathbf{S}^1 \times (-1, 1)$ .

We now return to spaces of curves.

**(7.4) Definitions.** For fixed  $\kappa_0 \in \mathbf{R}$  and  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ , let  $C$  denote the image of  $C_{\gamma}$  and  $D = -C$ . Assuming  $\gamma$  non-diffuse (meaning that  $C \cap D = \emptyset$ ), let  $\hat{C}$  (resp.  $\hat{D}$ ) be the connected component of  $\mathbf{S}^2 \setminus D$  containing  $C$  (resp. the component of  $\mathbf{S}^2 \setminus C$  containing  $D$ ) and let  $B = \hat{C} \cap \hat{D}$ .

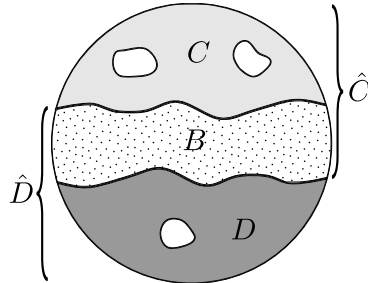


Figure 13: A sketch of the sets defined in (7.4) for a non-diffuse curve  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ . The lightly shaded region is  $C$  and the darkly shaded region is  $D = -C$ ; both are closed. The dotted region represents  $B$ , which is homeomorphic to  $\mathbf{S}^1 \times (-1, 1)$  by (7.5(c)).

**(7.5) Lemma.** *Let the notation be as in (7.4).*

- (a)  *$C$  and  $D$  are at a positive distance from each other.*
- (b)  *$B \subset \mathbf{S}^2 \setminus (C \cup D)$  is open and consists of all  $p \in \mathbf{S}^2$  such that: there exists a path  $\eta: [-1, 1] \rightarrow \mathbf{S}^2$  with*

$$\eta(-1) \in D, \quad \eta(1) \in C, \quad \eta(0) = p \quad \text{and} \quad \eta(-1, 1) \subset \mathbf{S}^2 \setminus (C \cup D).$$

- (c) *The set  $B$  is homeomorphic to  $\mathbf{S}^1 \times (-1, 1)$ .*

*Proof.* The proof of each item will be given separately.

- (a) This is clear, since  $C$  and  $D$  are compact sets which, by hypothesis, do not intersect.
- (b) Being components of open sets,  $\hat{C}$  and  $\hat{D}$  are open, hence so is  $B$ .

Suppose  $p \in B$ . Since  $p \in \hat{C}$ , there exists  $\eta_+: [0, 1] \rightarrow \mathbf{S}^2$  such that

$$\eta_+(0) = p, \quad \eta_+(1) \in C \quad \text{and} \quad \eta_+[0, 1] \subset \mathbf{S}^2 \setminus D.$$

We can actually arrange that  $\eta_+[0, 1] \subset \mathbf{S}^2 \setminus (C \cup D)$  by restricting the domain of  $\eta_+$  to  $[0, t_0]$ , where  $t_0 = \inf \{t \in [0, 1] : \eta_+(t) \in C\}$  and reparametrizing; note that  $t_0 > 0$  because  $B$  is open and disjoint from  $C$ . Similarly, there exists  $\eta_-: [-1, 0] \rightarrow \mathbf{S}^2$  such that

$$\eta_-(-1) \in D, \quad \eta_-(0) = p \quad \text{and} \quad \eta_-(-1, 0] \subset \mathbf{S}^2 \setminus (C \cup D).$$

Thus,  $\eta = \eta_- * \eta_+$  satisfies all the requirements stated in (b).

Conversely, suppose that such a path  $\eta$  exists. Then  $p \in \hat{C}$ , for there is a path  $\eta_+ = \eta|_{[0, 1]}$  joining  $p$  to a point of  $C$  while staying outside of  $D$  at all times. Similarly,  $p \in \hat{D}$ , whence  $p \in B$ .

- (c) The set  $\hat{C}$  is open and connected by definition. Its complement is also connected by (7.1(b)), as it consists of  $D$  and the components of  $\mathbf{S}^2 \setminus D$  distinct from  $\hat{C}$ . From (7.2(a)) it follows that  $\hat{C}$  is simply-connected. Further,  $\hat{C} \cap D = \emptyset$ , hence the complement of  $\hat{C}$  is non-empty and (7.2(b)) tells us that  $\hat{C}$  is homeomorphic to an open disk. By symmetry, the same is true of  $\hat{D}$ .

We claim that  $\hat{C} \cup \hat{D} = \mathbf{S}^2$ . To see this suppose  $p \notin C$ , and let  $A$  be the component of  $\mathbf{S}^2 \setminus C$  containing  $p$ . If  $A \cap D \neq \emptyset$  then  $A = \hat{D}$  by definition. Otherwise  $A \cap D = \emptyset$ , hence there exists a path in  $\mathbf{S}^2 \setminus D$

joining  $p$  to  $\partial A$ . By (7.1(a)),  $\partial A \subset C$ , consequently  $A \subset \hat{C}$ . In either case,  $p \in \hat{C} \cup \hat{D}$ .

We are thus in the setting of (7.3), and the conclusion is that

$$B = \hat{C} \cap \hat{D} \approx \mathbf{S}^1 \times (-1, 1). \quad \square$$

In what follows let  $\partial B_\gamma$  be the restriction of  $B_\gamma$  to  $[0, 1] \times \{0, \rho_0 - \pi\}$ , let

$$\hat{B} = \text{Im}(B_\gamma) \setminus \text{Im}(\partial B_\gamma),$$

and let

$$\bar{B}_\gamma: \mathbf{S}^1 \times [\rho_0 - \pi, 0] \rightarrow \mathbf{S}^2$$

be the unique map satisfying  $\bar{B}_\gamma \circ (\text{pr} \times \text{id}) = B_\gamma$ ,  $\text{pr}(t) = \exp(2\pi it)$ .

**(7.6) Lemma.** *Let  $\kappa_0 \in \mathbf{R}$  and suppose that  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  is non-diffuse. Then:*

- (a) *For any  $t \in [0, 1]$ ,  $B_\gamma(\{t\} \times (\rho_0 - \pi, 0))$  intersects  $B$ .*
- (b)  *$B \subset \hat{B}$ .*
- (c)  *$\bar{B}_\gamma^{-1}(q)$  is a finite set for any  $q \in \mathbf{S}^2$  and  $\bar{B}_\gamma: \bar{B}_\gamma^{-1}(\hat{B}) \rightarrow \hat{B}$  is a covering map.*

*Proof.* We split the proof into parts.

- (a) Note first that  $B_\gamma(t, 0) \in C$  and  $B_\gamma(t, \rho_0 - \pi) \in D$  for any  $t \in [0, 1]$  by definition. Let

$$\begin{aligned} \theta_1 &= \inf \{ \theta \in [\rho_0 - \pi, 0] : B_\gamma(t, \theta) \in C \}, \\ \theta_0 &= \sup \{ \theta \in [\rho_0 - \pi, \theta_1] : B_\gamma(t, \theta) \in D \}. \end{aligned}$$

Then  $\theta_0 < \theta_1$  by (7.5(a)). Let  $\eta = B_\gamma|_{\{t\} \times [\theta_0, \theta_1]}$ . Then

$$\eta(\theta_0) \in D, \quad \eta(\theta_1) \in C \quad \text{and} \quad \eta(\theta_0, \theta_1) \subset \mathbf{S}^2 \setminus (C \cup D)$$

by construction. Therefore, any point  $\eta(\theta)$  for  $\theta \in (\theta_0, \theta_1)$  satisfies the characterization of  $B$  given in (7.5(b)), and we conclude that

$$B_\gamma(\{t\} \times (\theta_0, \theta_1)) \subset B.$$

- (b) Let  $B_0 = B \cap \text{Im}(B_\gamma)$ . By part (a),  $B_0 \neq \emptyset$ . Since  $\text{Im}(\partial B_\gamma) \subset C \cup D$ , while  $B \cap (C \cup D) = \emptyset$  by definition,  $B \cap \text{Im}(\partial B_\gamma) = \emptyset$ . Hence,

$$B_0 = B \cap \bar{B}_\gamma(\mathbf{S}^1 \times (\rho_0 - \pi, 0)),$$

which is an open set because  $\bar{B}_\gamma$  is an immersion, by (4.7(a)). Since  $\text{Im}(B_\gamma)$  is compact,  $B_0$  is also closed in  $B$ . But  $B$  is connected by (7.5(c)), consequently  $B_0 = B$  and  $B \subset \hat{B}$ .

- (c) Let  $q \in \mathbf{S}^2$  be arbitrary. The set  $\bar{B}_\gamma^{-1}(q)$  is discrete because  $\bar{B}_\gamma$  is an immersion, and it is compact as a closed subset of  $\mathbf{S}^2$ . Hence, it must be finite. Now suppose  $q \in \hat{B}$ . Let  $\bar{B}_\gamma^{-1}(q) = \{p_i\}_{i=1}^n$  and choose disjoint open sets  $U_i \ni p_i$  restricted to which  $\bar{B}_\gamma$  is a diffeomorphism. Let  $U = \bigcup_{i=1}^n U_i$  and

$$W = \bar{B}_\gamma(U_1) \cap \cdots \cap \bar{B}_\gamma(U_n) \setminus \bar{B}_\gamma(\mathbf{S}^1 \times [\rho_0 - \pi, 0] \setminus U).$$

Then  $W$  is a distinguished neighborhood of  $q$ , in the sense that  $\bar{B}_\gamma^{-1}(W) = \bigsqcup_{i=1}^n V_i$  and  $\bar{B}_\gamma: V_i \rightarrow W$  is a diffeomorphism for each  $i$ , where

$$V_i = \bar{B}_\gamma^{-1}(W) \cap U_i. \quad \square$$

Parts (b) and (c) of (7.6) allow us to introduce a useful notion which essentially counts how many times a non-diffuse curve winds around  $\mathbf{S}^2$ .

**(7.7) Definition.** Let  $\kappa_0 \in \mathbf{R}$  and suppose that  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  is non-diffuse. We define the *rotation number*  $\nu(\gamma)$  of  $\gamma$  to be the number of sheets of the covering map  $\bar{B}_\gamma: \bar{B}_\gamma^{-1}(B) \rightarrow B$ .

*Remark.* Suppose now that  $\kappa_0 > 0$  and  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  is not only non-diffuse but also condensed (meaning that  $C$  is contained in a closed hemisphere). In this case, a “more natural” notion of the rotation number of  $\gamma$  is available, as described on p. 55. Let us temporarily denote by  $\bar{\nu}(\gamma)$  the latter rotation number. We claim that  $\bar{\nu}(\gamma) = \nu(\gamma)$  for any condensed and non-diffuse curve  $\gamma$ . It is easy to check that this holds whenever  $\gamma$  is a circle traversed a number of times. If  $\gamma_s$  ( $s \in [0, 1]$ ) is a continuous family of curves of this type then  $\nu(\gamma_s) = \nu(\gamma_0)$  and  $\bar{\nu}(\gamma_s) = \bar{\nu}(\gamma_0)$  for any  $s$ , since  $\nu$  and  $\bar{\nu}$  can only take on integral values and every element in their definitions depends continuously on  $s$ . Moreover, it follows from (6.2) that any condensed and non-diffuse curve is homotopic through curves of this type to a circle traversed a number of times.

**(7.8) Proposition.** *Let  $\kappa_0 \in \mathbf{R}$  and suppose that  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  is non-diffuse. Then there exists a constant  $K$  depending only on  $\kappa_0$  such that*

$$\text{tot}(\gamma) \leq K\nu(\gamma).$$

*Proof.* It is easy to check that being non-diffuse is an open condition. Using (2.8), we deduce that the closure of the subset of all  $C^2$  non-diffuse curves in  $\mathcal{L}_{\kappa_0}^{+\infty}$  contains the set of all (admissible) non-diffuse curves. Therefore, we lose no generality in restricting our attention to  $C^2$  curves.

Let  $b \in B$  be arbitrary; we have  $B = -B$ , hence  $-b \in B$  also. Let  $\hat{\gamma}$  be the other boundary curve of  $B_\gamma$ :

$$\hat{\gamma}(t) = B_\gamma(t, \rho_0 - \pi) = -\cos \rho_0 \gamma(t) - \sin \rho_0 \mathbf{n}(t) \quad (t \in [0, 1]).$$

Then

$$\hat{\gamma}'(t) = (\kappa(t) \sin \rho_0 - \cos \rho_0) \gamma'(t) = \frac{\sin(\rho_0 - \rho(t))}{\sin \rho(t)} \gamma'(t) \quad (t \in [0, 1]).^2 \quad (1)$$

(Here, as always,  $\kappa = \cot \rho$  is the geodesic curvature of  $\gamma$ .) In particular, the unit tangent vector  $\hat{\mathbf{t}}$  to  $\hat{\gamma}$  satisfies  $\hat{\mathbf{t}} = \mathbf{t}$ . By (2.21), the geodesic curvature  $\hat{\kappa}$  of  $\hat{\gamma}$  is given by

$$\hat{\kappa}(t) = \cot(\rho(t) - (\rho_0 - \pi)) = \cot(\rho(t) - \rho_0) \quad (t \in [0, 1]). \quad (2)$$

Define  $h, \hat{h}: [0, 1] \rightarrow (-1, 1)$  by

$$h(t) = \langle \gamma(t), b \rangle \quad \text{and} \quad \hat{h}(t) = \langle \hat{\gamma}(t), b \rangle. \quad (3)$$

These functions measure the “height” of  $\gamma$  and  $\hat{\gamma}$  with respect to  $\pm b$ . We cannot have  $|h(t)| = 1$  nor  $|\hat{h}(t)| = 1$  because the images of  $\gamma$  and  $\hat{\gamma}$  are contained in  $C$  and  $D$  respectively, which are disjoint from  $B$  (by definition (7.4)). Also,

$$h'(t) = |\gamma'(t)| \langle b, \mathbf{t}(t) \rangle, \quad \hat{h}'(t) = \frac{\sin(\rho_0 - \rho(t))}{\sin \rho(t)} h'(t). \quad (4)$$

Let  $\Gamma_t$  be the great circle whose center on  $\mathbf{S}^2$  is  $\mathbf{t}(t)$ ,

$$\Gamma_t = \{ \cos \theta \gamma(t) + \sin \theta \mathbf{n}(t) : \theta \in [-\pi, \pi) \}.$$

We have  $\gamma(t), \hat{\gamma}(t) \in \Gamma_t$  by definition. Moreover, the following conditions are equivalent:

- (i)  $b \in \Gamma_t$ .

<sup>2</sup>In this proof, derivatives with respect to  $t$  are denoted using a ' to simplify the notation.

(ii)  $h'(t) = 0$ .

(iii)  $\hat{h}'(t) = 0$ .

(iv) The segment  $B_\gamma(\{t\} \times (\rho_0 - \pi, 0))$  contains either  $b$  or  $-b$ .

The equivalence of the first three conditions follows from (4). The equivalence (i)  $\leftrightarrow$  (iv) follows from the facts that  $b \notin C \cap D$  and that  $\Gamma_t$  is the union of the segments  $\pm B_\gamma(\{t\} \times (\rho_0 - \pi, 0))$  and  $\pm C_\gamma(\{t\} \times [0, \rho_0])$  (see fig. 6, p. 37). The equivalence of the last three conditions tells us that  $h$  and  $\hat{h}$  have exactly  $2\nu(\gamma)$  critical points, for each of  $B_\gamma^{-1}(b)$  and  $B_\gamma^{-1}(-b)$  has cardinality  $\nu(\gamma)$ , by definition (7.7).

Suppose that  $\tau$  is a critical point of  $h$  and  $\hat{h}$ . Because  $b \in \Gamma_\tau \setminus (C \cup D)$ , we can write

$$b = \cos \theta \gamma(\tau) + \sin \theta \mathbf{n}(\tau), \text{ for some } \theta \in (\rho_0 - \pi, 0) \cup (\rho_0, \pi). \quad (5)$$

A straightforward calculation shows that:

$$h''(\tau) = \langle \gamma''(\tau), b \rangle = \frac{|\gamma'(\tau)|^2}{\sin \rho(\tau)} \sin(\theta - \rho(\tau)).$$

Using (5) and  $0 < \rho(\tau) < \rho_0$  we obtain that either

$$-\pi < \theta - \rho(\tau) < 0 \quad \text{or} \quad 0 < \theta - \rho(\tau) < \pi.$$

In any case, we deduce that  $h''(\tau) \neq 0$ . The proof that  $\tau$  is a nondegenerate critical point of  $\hat{h}$  is analogous: one obtains by another calculation that

$$\hat{h}''(\tau) = \frac{|\gamma'(\tau)|^2}{\sin^2(\rho(\tau))} \sin(\rho_0 - \rho(\tau)) \sin(\theta - \rho(\tau)),$$

and it follows from the above inequalities that  $\hat{h}''(\tau) \neq 0$ . In particular, two neighboring critical points  $\tau_0 < \tau_1$  of  $h$  (and  $\hat{h}$ ) cannot be both maxima or both minima for  $h$  (and  $\hat{h}$ ). We will prove the proposition by obtaining an upper bound for  $\text{tot}(\gamma|_{[\tau_0, \tau_1]})$ .

We first claim that  $B_\gamma|_{[\tau_0, \tau_1] \times [\rho_0 - \pi, 0]}$  is injective. Suppose for concreteness that  $h' < 0$  throughout  $(\tau_0, \tau_1)$  and that  $b = B_\gamma(\tau_0, \theta_0)$ ,  $-b = B_\gamma(\tau_1, \theta_1)$ , where  $\theta_0, \theta_1 \in (\rho_0 - \pi, 0)$ . Let  $\alpha = \alpha_1 * \alpha_2 * \alpha_3$  be the concatenation of the curves  $\alpha_i: [0, 1] \rightarrow \mathbf{S}^2$  given by

$$\begin{aligned} \alpha_1(t) &= B_\gamma(\tau_0, (1-t)\theta_0), & \alpha_2(t) &= \gamma((1-t)\tau_0 + t\tau_1), \\ \alpha_3(t) &= B_\gamma(\tau_1, t\theta_1), \end{aligned}$$



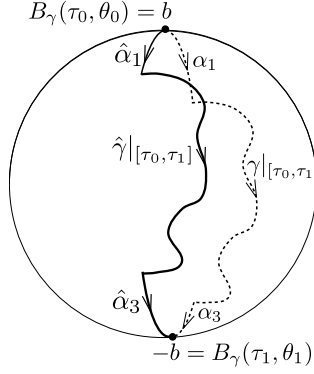


Figure 14: An illustration of the boundary of the rectangle  $R = B_\gamma|_{[\tau_0, \tau_1] \times [\rho_0 - \pi, 0]}$  considered in the proof of (7.8).

as sketched in fig. 14. Similarly, let  $\hat{\alpha}$  be the concatenation of the curves  $\hat{\alpha}_i: [0, 1] \rightarrow \mathbf{S}^2$ ,

$$\begin{aligned}\hat{\alpha}_1(t) &= B_\gamma(\tau_0, (1-t)\theta_0 + t(\rho_0 - \pi)), & \hat{\alpha}_2(t) &= \hat{\gamma}((1-t)\tau_0 + t\tau_1), \\ \hat{\alpha}_3(t) &= B_\gamma(\tau_1, (1-t)(\rho_0 - \pi) + t\theta_1).\end{aligned}$$

Define six functions  $h_i, \hat{h}_i: [0, 1] \rightarrow [-1, 1]$  by the formulas

$$h_i(t) = \langle \alpha_i(t), b \rangle \quad \text{and} \quad \hat{h}_i(t) = \langle \hat{\alpha}_i(t), b \rangle \quad (i = 1, 2, 3).$$

Note that  $h_2$  is essentially the restriction of  $h$  to  $[\tau_0, \tau_1]$  and similarly for  $\hat{h}_2$  (see (3)). Moreover, all of these functions are monotone decreasing. For  $i = 2$  this is immediate from (4) and the hypothesis that  $h' < 0$  on  $(\tau_0, \tau_1)$ . For  $i = 1, 3$  this follows from the fact that  $\alpha_i, \hat{\alpha}_i$  are geodesic arcs through  $\pm b$ , and our choice of orientations for these curves.

Because the map  $B_\gamma|_{[\tau_0, \tau_1] \times [\rho_0 - \pi, 0]}$  is an immersion, if  $B_\gamma$  is not injective then either  $\alpha$  and  $\hat{\alpha}$  intersect each other, or one of them has a self-intersection. We can discard the possibility that either curve has a self-intersection from the fact that all functions  $h_i, \hat{h}_i$  are monotone decreasing. Further, since  $B \approx \mathbf{S}^1 \times (-1, 1)$ , we can find a Jordan curve  $\beta: [0, 1] \rightarrow B$  through  $\pm b$  winding once around the  $\mathbf{S}^1$  factor. If  $\alpha$  and  $\hat{\alpha}$  intersect (at some point other than  $\alpha(0) = \hat{\alpha}(0)$  or  $\alpha(1) = \hat{\alpha}(1)$ ), then this must be an intersection of  $\gamma$  and  $\hat{\gamma}$ . This is impossible because  $\beta$ , which has image in  $B$ , separates  $C$  and  $D$ , which contain the images of  $\gamma$  and  $\hat{\gamma}$ , respectively.

Thus,  $R = B_\gamma|_{[\tau_0, \tau_1] \times [\rho_0 - \pi, 0]}$  is diffeomorphic to a rectangle, and its boundary consists of  $\hat{\gamma}|_{[\tau_0, \tau_1]}$ ,  $\gamma|_{[\tau_0, \tau_1]}$  (the latter with reversed orientation) and the two geodesic arcs  $B_\gamma(\{\tau_0\} \times [\rho_0 - \pi, 0])$  and  $B_\gamma(\{\tau_1\} \times [\rho_0 - \pi, 0])$ . Recall from (4.7) that  $\frac{\partial B_\gamma}{\partial t}$  is always orthogonal to  $\frac{\partial B_\gamma}{\partial \theta}$ . Using Gauss-Bonnet we deduce

that

$$\left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}\right) + \int_{\tau_0}^{\tau_1} \hat{\kappa}(t) |\hat{\gamma}'(t)| dt - \int_{\tau_0}^{\tau_1} \kappa(t) |\gamma'(t)| dt + \text{Area}(R) = 2\pi.$$

Using (1), (2) and the fact that  $\text{Area}(R) < \text{Area}(\mathbf{S}^2) = 4\pi$  we obtain:

$$\int_{\tau_0}^{\tau_1} \left( \cot \rho(t) + \frac{\sin(\rho_0 - \rho(t))}{\sin \rho(t)} \cot(\rho_0 - \rho(t)) \right) |\gamma'(t)| dt < 4\pi. \quad (6)$$

Let us see how this yields an upper bound for  $\text{tot}(\gamma|_{[\tau_0, \tau_1]})$ . From  $\cos(x) + \cos(y) = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$  and  $|\rho(t) - \frac{\rho_0}{2}| < \frac{\rho_0}{2}$  we deduce that

$$\begin{aligned} & \sin \rho(t) \left( \cot \rho(t) + \frac{\sin(\rho_0 - \rho(t))}{\sin \rho(t)} \cot(\rho_0 - \rho(t)) \right) \\ &= \cos \rho(t) + \cos(\rho_0 - \rho(t)) = 2 \cos\left(\frac{\rho_0}{2}\right) \cos\left(\rho(t) - \frac{\rho_0}{2}\right) \geq 2 \cos^2\left(\frac{\rho_0}{2}\right). \end{aligned}$$

The Euclidean curvature  $K$  of  $\gamma$  thus satisfies

$$\begin{aligned} K(t) &= \sqrt{1 + \kappa(t)^2} = \sqrt{1 + \cot^2 \rho(t)} = \csc \rho(t) \\ &\leq \frac{1}{2 \cos^2\left(\frac{\rho_0}{2}\right)} \left( \cot \rho(t) + \frac{\sin(\rho_0 - \rho(t))}{\sin \rho(t)} \cot(\rho_0 - \rho(t)) \right). \end{aligned} \quad (7)$$

Combining (6) and (7) we obtain:

$$\text{tot}(\gamma|_{[\tau_0, \tau_1]}) = \int_{\tau_0}^{\tau_1} K(t) |\gamma'(t)| dt < \frac{2\pi}{\cos^2\left(\frac{\rho_0}{2}\right)}.$$

Extending  $\gamma$  to all of  $\mathbf{R}$  by declaring it to be 1-periodic and choosing consecutive critical points  $\tau_0 < \tau_1 < \dots < \tau_{2\nu(\gamma)-1} < \tau_{2\nu(\gamma)}$ , so that  $\tau_{2\nu(\gamma)} = \tau_0 + 1$ , we finally conclude from the previous estimate (with  $[\tau_{i-1}, \tau_i]$  in place of  $[\tau_0, \tau_1]$ ) that

$$\text{tot}(\gamma) = \sum_{i=1}^{2\nu(\gamma)} \text{tot}(\gamma|_{[\tau_{i-1}, \tau_i]}) < \frac{4\pi}{\cos^2\left(\frac{\rho_0}{2}\right)} \nu(\gamma). \quad \square$$

## 8

### Homotopies of Circles

Let  $k \geq 1$  be an integer. The *bending of the  $k$ -equator* is an explicit homotopy (to be defined below) from a great circle traversed  $k$  times to a great circle traversed  $k + 2$  times. It is an “optimal” homotopy of this type, in the following sense: It is possible to deform a circle traversed  $k$  times into a circle traversed  $k + 2$  times in  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$  if and only if we may carry out the bending of the  $k$ -equator in this space (meaning that the absolute value of the geodesic curvature is bounded by  $\kappa_1$  throughout the bending). A special case of this construction was considered by Saldanha in [12].

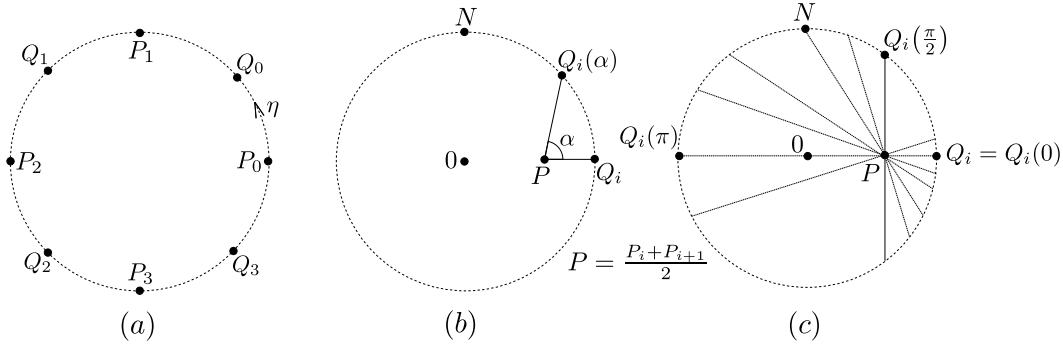


Figure 15:

Let  $N = (0, 0, 1) \in \mathbf{S}^2$  be the north pole, let

$$\eta(t) = (\cos(2k\pi t), \sin(2k\pi t), 0) \quad (t \in [0, 1])$$

be a parametrization of the equator traversed  $k \geq 1$  times ( $k \in \mathbf{N}$ ) and let

$$P_i = \eta\left(\frac{i}{2k+2}\right), \quad Q_i = \eta\left(\frac{i+\frac{1}{2}}{2k+2}\right) \quad (i = 0, 1, \dots, 2k+1),$$

as illustrated in fig. 15(a) for  $k = 1$ . Define  $Q_i(\alpha)$  (see fig. 15(b)) to be the unique point in the geodesic through  $N$  and  $Q_i$  such that

$$\angle Q_i\left(\frac{P_i + P_{i+1}}{2}\right) Q_i(\alpha) = \alpha \quad (-\pi \leq \alpha \leq \pi, \quad i = 0, 1, \dots, 2k+1).$$

Let  $A_i(\alpha) \subset \mathbf{S}^2$  be the arc of circle through  $P_i Q_i(\alpha) P_{i+1}$ , with orientation determined by this ordering of the three points, and define

$$\sigma_{\alpha,i}: \left[0, \frac{1}{2k+2}\right] \rightarrow \mathbf{S}^2 \quad (0 \leq \alpha \leq \pi, i = 0, \dots, 2k+1)$$

to be a parametrization of  $A_i((-1)^i \alpha)$  by a multiple of arc-length, as illustrated in fig. 16 below for  $k = 1$ . Note that  $A_i(0)$  is just  $\frac{k}{2k+2}$  of the equator, while  $A_i(\pi)$  is the “complement” of  $A_i(0)$ , which is  $\frac{k+2}{2k+2}$  of the equator.

Let  $\sigma_\alpha: [0, 1] \rightarrow \mathbf{S}^2$  be the concatenation of all the  $\sigma_{\alpha,i}$ , for  $i$  increasing from 0 to  $2k+1$  (as in fig. 16). Then  $\sigma_0$  is the equator traversed  $k$  times, while  $\sigma_\pi$  is the equator traversed  $k+2$  times, in the opposite direction. The curve  $\sigma_\alpha$  is closed and regular for all  $\alpha \in [0, \pi]$ . However, its geodesic curvature is a step function, taking the value  $(-1)^i \kappa(\alpha)$  for  $t \in (\frac{i}{2k+2}, \frac{i+1}{2k+2})$ , where  $\kappa(\alpha)$  depends only on  $\alpha$ . At the points  $t = \frac{i}{2k+2}$  the curvature is not defined, except for  $\alpha = 0, \pi$ , when the curvature vanishes identically.

We are only interested in the maximum value of  $\kappa(\alpha)$  for  $0 \leq \alpha \leq \pi$ , which can be easily determined. For any  $\alpha$ , the center of the circle  $C$  of which  $A_i(\alpha)$  is an arc is contained in the plane  $\Pi_1$  through  $0$ ,  $Q_i$  and  $N$ , since this plane is the locus of points equidistant from  $P_i$  and  $P_{i+1}$  ( $\Pi_1$  is the plane of figures 15(b) and 15(c)). By definition,  $C$  is contained in the plane  $\Pi_2$  through  $P_i$ ,  $Q_i(\alpha)$  and  $P_{i+1}$ . Thus, the center of  $C$  lies in the line  $\Pi_1 \cap \Pi_2 = PQ_k(\alpha)$ , and the segment of this line bounded by  $\mathbf{S}^2$  is a diameter of  $C$ . Clearly, this diameter is shortest when  $\alpha = \frac{\pi}{2}$  (see fig. 15(c)). (More precisely, the shortest chord through a point lying in the interior of a circle is the one which is perpendicular to the diameter through this point; the proof is an exercise in elementary geometry.) The corresponding spherical radius is  $\rho = \frac{k\pi}{2k+2}$ , hence the maximum value attained by  $\kappa(\alpha)$  for  $0 \leq \alpha \leq \pi$  is

$$\kappa\left(\frac{\pi}{2}\right) = \cot\left(\frac{k\pi}{2k+2}\right) = \tan\left(\frac{\pi}{2k+2}\right),$$

and the minimum value is  $-\kappa\left(\frac{\pi}{2}\right)$ .

**(8.1) Definition.** Let  $\sigma_\alpha$  be as in the discussion above ( $0 \leq \alpha \leq \pi$ ) and assume that

$$\kappa_1 > \tan\left(\frac{\pi}{2k+2}\right). \quad (1)$$

The *bending of the  $k$ -equator* is the family of curves  $\eta_s \in \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$  given by:

$$\eta_s(t) = (\Phi_{\sigma_{s\pi}}(0))^{-1} \sigma_{s\pi}(t) \quad (s, t \in [0, 1]).$$

Note that  $\eta_0$  is the equator of  $\mathbf{S}^2$  traversed  $k$  times and  $\eta_1$  is the equator traversed  $k+2$  times, in the same direction. The following result is an

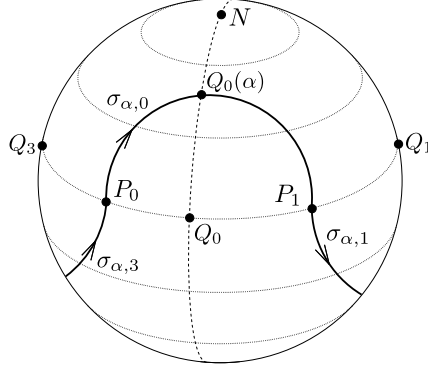


Figure 16: An illustration of the bending of the 1-equator. The curve  $\sigma_\alpha$  is the concatenation of  $\sigma_{\alpha,0}, \dots, \sigma_{\alpha,3}$ .

immediate consequence of the discussion above.

**(8.2) Proposition.** *Let  $\kappa_0 = \cot \rho_0 \in \mathbf{R}$  and let  $\sigma_k, \sigma_{k+2} \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  be circles traversed  $k$  and  $k+2$  times, respectively. Then  $\sigma_k$  lies in the same component of  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$  as  $\sigma_{k+2}$  if*

$$k \geq \left\lfloor \frac{\pi}{\rho_0} \right\rfloor. \quad (2)$$

*Proof.* Let  $\rho_1 = \frac{\pi - \rho_0}{2}$ , so that  $\kappa_1 = \cot \rho_1$  satisfies (1). Let  $\gamma_s$  ( $s \in [0, 1]$ ) be the image of the bending  $\eta_s$  of the  $k$ -equator under the homeomorphism  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I) \approx \mathcal{L}_{\kappa_0}^{+\infty}(I)$  of (2.26). Then  $\gamma_0$  is some circle traversed  $k$  times, while  $\gamma_1$  is a circle traversed  $k+2$  times. Using (4.4) we deduce that  $\sigma_k \simeq \gamma_0 \simeq \gamma_1 \simeq \sigma_{k+2}$ , hence  $\sigma_k$  and  $\sigma_{k+2}$  lie in the same component of  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ .  $\square$

**(8.3) Corollary.** *Let  $\rho_i = \operatorname{arccot}(\kappa_i)$ ,  $i = 1, 2$ , and suppose that  $\rho_1 - \rho_2 > \frac{\pi}{2}$ . Let  $\sigma_{k_0}, \sigma_{k_1} \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  (resp.  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ ) be two parametrized circles traversed  $k_0$  and  $k_1$  times, respectively. Then  $\sigma_{k_0}$  and  $\sigma_{k_1}$  lie in the same connected component if and only if  $k_0 \equiv k_1 \pmod{2}$ .*

*Proof.* Under the homeomorphism  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I) \approx \mathcal{L}_{\kappa_0}^{+\infty}(I)$  of (2.25), the condition  $\rho_1 - \rho_2 > \frac{\pi}{2}$  translates into  $\rho_0 > \frac{\pi}{2}$ . The result is an immediate consequence of (2.15), (4.4) and (8.2).  $\square$

### Homotopies of condensed curves

The previous corollary settles the question of when two circles in  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$  lie in the same component for  $\kappa_0 < 0$ . Because of this, we will assume for the rest of the section that  $\kappa_0 \geq 0$ ; the following proposition implies the converse to (8.2), and together with it, settles the same question in this case.

**(8.4) Proposition.** *Let  $\kappa_0 = \cot \rho_0 \geq 0$  and let*

$$n = \left\lfloor \frac{\pi}{\rho_0} \right\rfloor + 1.$$

*Suppose that  $s \mapsto \gamma_s \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  is a homotopy, with  $\gamma_0$  condensed and  $\nu(\gamma_0) \leq n - 2$  ( $s \in [0, 1]$ ). Then  $\gamma_s$  is condensed and  $\nu(\gamma_s) = \nu(\gamma_0)$  for all  $s \in [0, 1]$ .*

In particular, taking  $\gamma_0$  to be a circle  $\sigma_k$  traversed  $k$  times for  $k \leq n - 2$ , we conclude that it is not possible to deform  $\sigma_k$  into a circle traversed  $k + 2$  times in  $\mathcal{L}_{\kappa_0}^{+\infty}$ . The proof of (8.4) will be broken into several parts. We start with the definition of an equatorial curve, which is just a borderline case of a condensed curve.

**(8.5) Definition.** Let  $\kappa_0 \geq 0$ . We shall say that a curve  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  is *equatorial* if the image  $C$  of its caustic band is contained in a closed hemisphere, but not in any open hemisphere. Let

$$H_\gamma = \{p \in \mathbf{S}^2 : \langle p, h_\gamma \rangle \geq 0\}$$

be a closed hemisphere containing  $\gamma$ , and let

$$E_\gamma = \{p \in \mathbf{S}^2 : \langle p, h_\gamma \rangle = 0\}$$

denote the corresponding *equator*. Also, let  $\tilde{\gamma}: [0, 1] \rightarrow \mathbf{S}^2$  be the curve given by

$$\tilde{\gamma}(t) = C_\gamma(t, \rho_0).$$

**(8.6) Lemma.** *Let  $\kappa_0 \geq 0$ , let  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  be an equatorial curve of class  $C^2$ . Then:*

- (a) *The hemisphere  $H_\gamma$  and the equator  $E_\gamma$  defined above are uniquely determined by  $\gamma$ .*
- (b) *The geodesic curvature  $\tilde{\kappa}$  of  $\tilde{\gamma}$  is given by:*

$$\tilde{\kappa} = \cot(\rho_0 - \rho) > 0.$$

*Proof.* Suppose that  $C = \text{Im}(C_\gamma)$  is contained in distinct closed hemispheres  $H_1$  and  $H_2$ . Then it is contained in the closed lune  $H_1 \cap H_2$ . Since the curves  $\gamma, \tilde{\gamma}$ , whose images form the boundary of  $C$ , have a unit tangent vector at all points, they cannot pass through either of the points in  $E_1 \cap E_2$  (where  $E_i$  is

the equator corresponding to  $H_i$ ). It follows that  $C$  is contained in an open hemisphere, a contradiction which establishes (a).

For part (b) we calculate:<sup>1</sup>

$$\check{\gamma}'(t) = |\gamma'(t)| (\cos \rho_0 - \kappa(t) \sin \rho_0) \mathbf{t}(t) \quad (3)$$

$$\check{\gamma}''(t) = |\gamma'(t)|^2 (\cos \rho_0 - \kappa(t) \sin \rho_0) (-\gamma(t) + \kappa(t) \mathbf{n}(t)) + \lambda(t) \mathbf{t}(t), \quad (4)$$

where  $\kappa$ ,  $\mathbf{t}$  and  $\mathbf{n}$  denote the geodesic curvature of and unit and normal vectors to  $\gamma$ , respectively, and the value of  $\lambda(t)$  is irrelevant to us. Hence,

$$\check{\kappa} = \frac{\langle \check{\gamma}, \check{\gamma}' \times \check{\gamma}'' \rangle}{|\check{\gamma}'|^3} = \frac{\kappa \cos \rho_0 + \sin \rho_0}{|\cos \rho_0 - \kappa \sin \rho_0|} = \frac{\cos(\rho_0 - \rho)}{|\sin(\rho - \rho_0)|} = \cot(\rho_0 - \rho). \quad \square$$

**(8.7) Lemma.** *Let  $\kappa_0 \geq 0$  and  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  be an equatorial curve of class  $C^2$ . Take  $N \in E_\gamma$  and define  $h, \check{h}: [0, 1] \rightarrow \mathbf{R}$  by*

$$h(t) = \langle \gamma(t), N \rangle, \quad \check{h}(t) = \langle \check{\gamma}(t), N \rangle. \quad (5)$$

(a) *The following conditions are equivalent:*

- (i)  $\pm N \in \Gamma_\tau$  for some  $\tau \in [0, 1]$ .
- (ii)  $\tau \in [0, 1]$  is a critical point of  $h$ .
- (iii)  $\tau \in [0, 1]$  is a critical point of  $\check{h}$ .

(b) *If  $\tau$  is a common critical point of  $h, \check{h}$ , then  $h''(\tau)\check{h}''(\tau) < 0$ .*

(c) *If  $\tau < \bar{\tau}$  are neighboring critical points then  $h''(\tau)h''(\bar{\tau}) < 0$  and  $\check{h}''(\tau)\check{h}''(\bar{\tau}) < 0$ .*

Recall that  $\Gamma_t$  is the great circle

$$\Gamma_t = \{ \cos \theta \gamma(t) + \sin \theta \mathbf{n}(t) : \theta \in [-\pi, \pi) \}.$$

Part (b) implies in particular that all critical points of  $h, \check{h}$  are nondegenerate.

*Proof.* A straightforward calculation using (3) shows that:

$$h'(t) = |\gamma'(t)| \langle N, \mathbf{t}(t) \rangle, \quad \check{h}'(t) = \frac{\sin(\rho(t) - \rho_0)}{\sin \rho(t)} h'(t) \quad (t \in [0, 1]). \quad (6)$$

The equivalence of the conditions in (a) is immediate from this and the definition of  $\Gamma_t$ .

<sup>1</sup>For the rest of the section we denote derivatives with respect to  $t$  by a  $'$  to unclutter the notation.

From  $\pm N \in E_\gamma$  and  $C = \text{Im}(C_\gamma) \subset H_\gamma$ , it follows that  $\pm N \notin C([0, 1] \times (0, \rho_0))$ . Thus, if  $\tau$  is a critical point of  $h, \check{h}$ , i.e., if  $N \in \Gamma_\tau$  then we can write

$$N = \cos \theta \gamma(\tau) + \sin \theta \mathbf{n}(\tau) \quad \text{for some } \theta \in [\rho_0 - \pi, 0] \cup [\rho_0, \pi]. \quad (7)$$

Another calculation, with the help of (4), yields:

$$h''(\tau) = \frac{|\gamma'(\tau)|^2}{\sin \rho(\tau)} \sin(\theta - \rho(\tau)), \quad \check{h}''(\tau) = \frac{|\gamma'(\tau)|^2}{\sin^2 \rho(\tau)} \sin(\theta - \rho(\tau)) \sin(\rho(\tau) - \rho_0)$$

Taking the possible values for  $\theta$  in (7) and  $0 < \rho(\tau) < \rho_0$  into account, we deduce that

$$h''(\tau)\check{h}''(\tau) = \frac{|\gamma'(\tau)|^4}{\sin^3 \rho(\tau)} \sin^2(\theta - \rho(\tau)) \sin(\rho(\tau) - \rho_0) < 0,$$

since all terms here are positive except for  $\sin(\rho(\tau) - \rho_0)$ . This proves (b).

For part (c), suppose that  $\tau < \bar{\tau}$  are neighboring critical points, but  $h''(\tau)h''(\bar{\tau}) > 0$ . This means that  $h'$  vanishes at  $\tau, \bar{\tau}$  and takes opposite signs on the intervals  $(\tau, \tau + \varepsilon)$  and  $(\bar{\tau} - \varepsilon, \bar{\tau})$  for small  $\varepsilon > 0$ . Hence, it must vanish somewhere in  $(\tau, \bar{\tau})$ , a contradiction. The proof for  $\check{h}$  is the same.  $\square$

Let  $\kappa_0 \geq 0$ ,  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  be an equatorial curve and  $\text{pr}: \mathbf{S}^2 \rightarrow \mathbf{R}^2$  denote the stereographic projection from  $-h_\gamma$ , where  $H_\gamma = \{p \in \mathbf{S}^2 : \langle p, h_\gamma \rangle \geq 0\}$ . As for any condensed curve, we may define a (non-unique) continuous angle function  $\theta$  by the formula:

$$\exp(i\theta(t)) = \mathbf{t}_\eta(t), \quad \eta(t) = \text{pr} \circ \gamma(t) \quad (t \in [0, 1]);$$

here  $\mathbf{t}_\eta$  is the unit tangent vector, taking values in  $\mathbf{S}^1$ , of the plane curve  $\eta$ . The function  $\theta$  is strictly decreasing since  $\kappa_0 \geq 0$ , and

$$2\pi\nu(\gamma) = \theta(0) - \theta(1).$$

**(8.8) Lemma.** *Let  $\kappa_0 \geq 0$ ,  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  be an equatorial curve of class  $C^2$  and*

$$n = \left\lfloor \frac{\pi}{\rho_0} \right\rfloor + 1.$$

*Then  $\nu(\gamma) \geq n - 1$ .*

*Proof.* Let  $C = \text{Im}(C_\gamma)$ ,  $H = H_\gamma$  be the closed hemisphere containing  $\gamma$  and  $E = E_\gamma$  be the corresponding equator, oriented so that  $H$  lies to its left. It



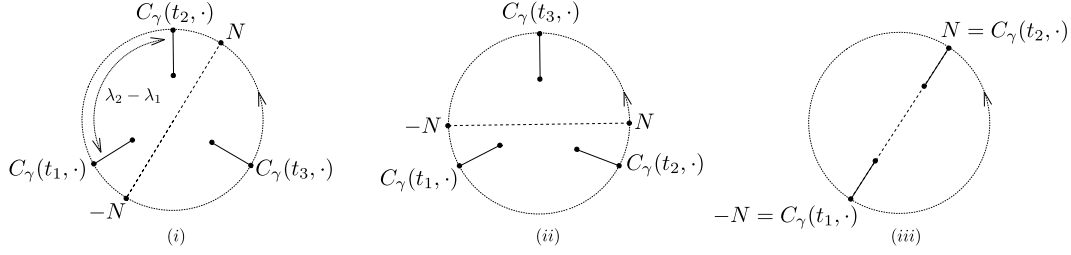


Figure 17: Three possibilities for an equatorial curve  $\gamma$ . The circle represents  $E_\gamma$  and its interior represents  $H_\gamma$ , seen from above.

follows from the combination of (11.1), (11.5) and (11.2) that either we can find two antipodal points in  $C \cap E$  or we can choose  $t_1 < t_2 < t_3$  and  $\theta_i \in \{0, \rho_0\}$  such that 0 is a convex combination of the points  $C_\gamma(t_i, \theta_i) \in C \cap E$ . There are three possibilities, as depicted in fig. 17; the only difference between the first two is the order of the points in the orientation of  $E$ .

In cases (i) and (ii), choose  $N$  in  $E$  so that

$$\langle C_\gamma(t_2, \theta_2), N \rangle = -\langle C_\gamma(t_1, \theta_1), N \rangle > 0.$$

Let  $h$  and  $\check{h}$  be as in (5) and define latitude functions  $\lambda, \check{\lambda}$  by

$$\lambda(t) = \arcsin(h(t)), \quad \check{\lambda}(t) = \arcsin(\check{h}(t)) \quad (t \in [0, 1]).$$

Let  $\tau_1 < \dots < \tau_{k_1}$  be all the common critical points of these functions in the interval  $[t_1, t_2)$ , and let

$$m_j = \min\{\lambda(\tau_j), \check{\lambda}(\tau_j)\}, \quad M_j = \max\{\lambda(\tau_j), \check{\lambda}(\tau_j)\}.$$

From (8.7(a)), we deduce that

$$M_j - m_j = \rho_0 \quad \text{for all } j = 1, \dots, k_1, \quad (8)$$

while from (8.7(b)) and (8.7(c)), we deduce that the  $\tau_j$  are alternatingly maxima and minima of  $\lambda$  (resp. minima and maxima of  $\check{\lambda}$ ) as  $j$  goes from 1 to  $k_1$ , whence

$$M_j > m_{j+1} \quad \text{for all } j = 1, \dots, k_1 - 1. \quad (9)$$

Let

$$\lambda_2 = \max\{\lambda(t_2), \check{\lambda}(t_2)\} \quad \text{and} \quad \lambda_1 = \min\{\lambda(t_1), \check{\lambda}(t_1)\} = -\lambda_2.$$

Then  $\lambda_2 - \lambda_1$  is just the angle between  $C_\gamma(t_1, \cdot) \cap E$  and  $C_\gamma(t_2, \cdot) \cap E$  measured

along  $E$ , as depicted in fig. 17(i). For the rest of the proof we consider each case separately.

In case (i),

$$m_1 \leq \lambda_1 \quad \text{and} \quad \lambda_2 \leq M_{k_1}. \quad (10)$$

Combining (8), (9) and (10), we find that

$$k_1 \rho_0 = \sum_{j=1}^{k_1} (M_j - m_j) > \sum_{j=1}^{k_1-1} (m_{j+1} - m_j) + M_{k_1} - m_{k_1} = M_{k_1} - m_1 \geq \lambda_2 - \lambda_1. \quad (11)$$

Let there be  $k_2$  (resp.  $k_3$ ) critical points of  $h, \check{h}$  in the interval  $[t_2, t_3]$  (resp.  $[t_3, t_1 + 1]$ ), where for the latter we are considering  $\gamma$  as a 1-periodic curve. Then an analogous result to (11) holds for  $k_2$  and  $k_3$ , and summing all three inequalities we conclude that

$$k_1 + k_2 + k_3 > \frac{2\pi}{\rho_0} \geq 2(n - 1).$$

In case (i), the number of half-turns of the tangent vector to the image of  $\gamma$  under stereographic projection through  $-h_\gamma$  in  $[0, 1]$  is given by  $k_1 + k_2 + k_3 - 2$ . Hence,

$$\nu(\gamma) = \frac{k_1 + k_2 + k_3 - 2}{2} > n - 2,$$

as claimed.

In case (ii), a direct calculation using basic trigonometry shows that

$$\begin{aligned} m_1 &< \arcsin(\cos \rho_0 \sin \lambda_1) = -\arcsin(\cos \rho_0 \sin \lambda_2) \\ \text{and } M_{k_1} &> \arcsin(\cos \rho_0 \sin \lambda_2). \end{aligned}$$

Combining this with (8) and (9), we obtain that

$$\begin{aligned} k_1 \rho_0 &= \sum_{j=1}^{k_1} (M_j - m_j) > \sum_{j=1}^{k_1-1} (m_{j+1} - m_j) + M_{k_1} - m_{k_1} \\ &= M_{k_1} - m_1 > 2 \arcsin(\cos \rho_0 \sin \lambda_2), \end{aligned}$$

and similarly for  $k_2$  and  $k_3$ , where the latter denote the number of critical points of  $h, \check{h}$  in the intervals  $[t_2, t_3]$  and  $[t_3, t_1 + 1]$ , respectively. More precisely, we have

$$k_1 + k_2 + k_3 > \frac{2}{\rho_0} \sum_{i=1}^3 \arcsin(\cos \rho_0 \sin \lambda_{2i}), \quad (12)$$

where  $\lambda_4 = \max \{ \lambda(t_3), \check{\lambda}(t_3) \}$ ,  $\lambda_6 = \max \{ \lambda(t_1), \check{\lambda}(t_1) \}$  and these latitudes are measured with respect to the chosen points  $\pm N$  corresponding to each of the intervals  $[t_2, t_3]$  and  $[t_3, t_3 + 1]$ . In case (ii), the number of half-turns of the

tangent vector to the image of  $\gamma$  under stereographic projection through  $-h_\gamma$  in  $[0, 1]$  is given by  $k_1 + k_2 + k_3 - 2$ . Hence, it follows from (12) and lemma (8.9) below that

$$\nu(\gamma) = \frac{k_1 + k_2 + k_3 + 2}{2} > \left(\frac{\pi}{\rho_0} - 2\right) + 1 \geq n - 2,$$

as we wished to prove.

Finally, in case (iii), we may choose  $\pm N \in E \cap C$ , that is, we may find  $t_1 < t_2$  and  $\theta_i \in \{0, \rho_0\}$  such that

$$N = C_\gamma(t_2, \theta_2) = -C_\gamma(t_1, \theta_1).$$

In this case  $\lambda_2 - \lambda_1 = \pi$  and

$$\nu(\gamma) = \frac{k_1 + k_2 - 2}{2},$$

where  $k_1$  (resp.  $k_2$ ) is the number of critical points of  $h, \check{h}$  in  $[t_1, t_2]$  (resp.  $[t_2, t_1 + 1]$ ). Note that  $t_1, t_2$  are critical points of  $h$  which are counted twice in the sum  $k_1 + k_2$  (under the identification of  $t_1$  with  $t_1 + 1$ ); this is the reason why we need to subtract 2 from  $k_1 + k_2$  to calculate the number of half-turns of the tangent vector. Using (9) one more time, we deduce that

$$k_1 \rho_0 = \sum_{j=1}^{k_1} (M_j - m_j) > \sum_{j=1}^{k_1-1} (m_{j+1} - m_j) + M_{k_1} - m_{k_1} = M_{k_1} - m_1 = \lambda_2 - \lambda_1 = \pi;$$

similarly,  $k_2 \rho_0 > \pi$ . Therefore,

$$\nu(\gamma) = \frac{k_1 + k_2 - 2}{2} > \frac{\pi}{\rho_0} - 1 \geq n - 2. \quad \square$$

Here is the technical lemma that was invoked in the proof of (8.8).

**(8.9) Lemma.** *Let  $\lambda_2 + \lambda_4 + \lambda_6 = \pi$ ,  $0 \leq \lambda_i \leq \frac{\pi}{2}$  and  $0 < \rho_0 \leq \frac{\pi}{2}$ . Then*

$$\arcsin(\cos \rho_0 \sin \lambda_2) + \arcsin(\cos \rho_0 \sin \lambda_4) + \arcsin(\cos \rho_0 \sin \lambda_6) \geq \pi - 2\rho_0$$

*Proof.* Let  $f: [0, \pi] \rightarrow \mathbf{R}$  be the function given by  $f(t) = \arcsin(\cos \rho_0 \sin t)$ . Then

$$f''(t) = -\frac{\sin^2 \rho_0 \cos \rho_0 \sin t}{(1 - \cos^2 \rho_0 \sin^2 t)^{\frac{3}{2}}},$$

so that  $f''(t) \leq 0$  for all  $t \in (0, \pi)$  and  $f$  is a concave function. Consequently,

$$f(s_1 a + s_2 b + s_3 c) \geq s_1 f(a) + s_2 f(b) + s_3 f(c) \quad (13)$$

for any  $a, b, c \in [0, \pi]$ ,  $s_i \in [0, 1]$ ,  $s_1 + s_2 + s_3 = 1$ . Define  $g: T \rightarrow \mathbf{R}$  by  $g(x, y, z) = f(x) + f(y) + f(z)$ , where

$$T = \{(x, y, z) \in \mathbf{R}^3 : x + y + z = \pi, \ x, y, z \in [0, \frac{\pi}{2}]\}.$$

In other words,  $T$  is the triangle with vertices  $A = (0, \frac{\pi}{2}, \frac{\pi}{2})$ ,  $B = (\frac{\pi}{2}, 0, \frac{\pi}{2})$  and  $C = (\frac{\pi}{2}, \frac{\pi}{2}, 0)$ . It follows from (13) (applied three times) that

$$g(s_1u + s_2v + s_3w) \geq s_1g(u) + s_2g(v) + s_3g(w) \quad (14)$$

for any  $u, v, w \in T$ ,  $s_i \in [0, 1]$ ,  $s_1 + s_2 + s_3 = 1$ . Moreover, a direct verification shows that

$$g(A) = g(B) = g(C) = 2 \arcsin(\cos \rho_0) = \pi - 2\rho_0.$$

If  $p \in T$  then we can write

$$p = s_1A + s_2B + s_3C \text{ for some } s_1, s_2, s_3 \in [0, 1] \text{ with } s_1 + s_2 + s_3 = 1.$$

Therefore, (14) guarantees that

$$g(p) \geq s_1g(A) + s_2g(B) + s_3g(C) = \pi - 2\rho_0. \quad \square$$

*Proof of (8.4).* If  $\gamma_s$  is condensed for all  $s \in [0, 1]$ , then  $s \mapsto \nu(\gamma_s)$  is defined and constant, since it can only take on integral values. Thus, if the assertion is false, there must exist  $s \in [0, 1]$ , say  $s = 1$ , such that  $\gamma_s$  is not condensed. By (6.1),  $\gamma_0$  is homotopic to a circle traversed  $\nu(\gamma_0)$  times. Moreover, the set of non-condensed curves is open. Together with (2.10), this shows that there exist  $C^2$  curves  $\gamma_{-1}, \gamma_2$  such that:

- (i) There exist a path joining  $\gamma_{-1}$  to  $\gamma_0$  and a path joining  $\gamma_1$  to  $\gamma_2$  in  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ ;
- (ii)  $\gamma_{-1}$  is condensed and has rotation number  $\nu(\gamma_0)$ ;
- (iii)  $\gamma_2$  is not condensed.

Consider the map  $f: \mathbf{S}^0 \rightarrow \mathcal{L}_{\kappa_0}^{+\infty}(I)$  given by  $f(-1) = \gamma_{-1}$ ,  $f(1) = \gamma_2$ . The existence of the homotopy  $\gamma_s$  ( $s \in [0, 1]$ ) tells us that  $f$  is nullhomotopic in  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ . By (2.10),  $f$  must be nullhomotopic in  $\mathcal{C}_{\kappa_0}^{+\infty}(I)$ . In other words, we may assume at the outset that each  $\gamma_s$  is of class  $C^2$  ( $s \in [0, 1]$ ).

With this assumption in force, let  $s_0$  be the infimum of all  $s \in [0, 1]$  such that  $\gamma_s$  is not condensed, and let  $\gamma = \gamma_{s_0}$ . Then  $\gamma$  must be condensed by (11.2), and it must be equatorial by our choice of  $s_0$ . In addition,  $\nu(\gamma_s)$  must be

constant ( $s \in [0, s_0]$ ), since it can only take on integral values. This contradicts (8.8).  $\square$

## 9

### Proofs of the Main Theorems

We will now collect some of the results from the previous sections in order to prove the theorems stated in §4. We repeat their statements here for convenience.

**(3.2) Theorem.** *Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ . Every curve in  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  (resp.  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ ) lies in the same component as a circle traversed  $k$  times, for some  $k \in \mathbf{N}$  (depending on the curve).*

*Proof.* By the homeomorphism  $\mathcal{L}_{\kappa_1}^{\kappa_2} \approx \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  of (2.15), it does not matter whether we prove the theorem for  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  or for  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ . Further, by (2.25), it suffices to consider spaces of type  $\mathcal{L}_{\kappa_0}^{+\infty}$ , for  $\kappa_0 \in \mathbf{R}$ . If  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  is diffuse, then it is homotopic to a circle by (5.11). If it is condensed, then the same conclusion holds by (6.1) (when  $\kappa_0 > 0$ ), (6.7) (when  $\kappa_0 < 0$ ) and Little's theorem (when  $\kappa_0 = 0$ ).

Assume then that  $\gamma$  is neither homotopic to a condensed nor to a diffuse curve. Since  $\gamma$  itself is non-condensed by hypothesis, (5.12) guarantees that we may find  $\varepsilon > 0$  and a chain of grafts  $(\gamma_s)$  with  $\gamma_0 = \gamma$  and  $\gamma_s \in \mathcal{L}_{\kappa_0}^{+\infty}$  for all  $s \in [0, \varepsilon)$ . Let  $(\gamma_s)$ ,  $s \in J$ , be a maximal chain of grafts starting at  $\gamma = \gamma_0$ , where  $J$  is an interval of type  $[0, \sigma)$  or  $[0, \sigma]$ . That such a chain exists follows by a straightforward argument involving Zorn's lemma, since the grafting relation is an equivalence relation, as proved in (5.6).<sup>1</sup> By hypothesis, no curve  $\gamma_s$  is diffuse, hence  $\nu(\gamma_s)$  is well-defined and independent of  $s$ , and (7.8) yields that  $\sigma < +\infty$ . If the interval is of the first type, then we obtain a contradiction from (5.7), and if the interval is closed, then we can apply (5.12) to  $\gamma_\sigma$  to extend the chain, again contradicting the choice of  $J$ . We conclude that  $\gamma$  must be homotopic either to a condensed or to a diffuse curve. In any case,  $\gamma$  is homotopic in  $\mathcal{L}_{\kappa_0}^{+\infty}$  to a circle traversed a number of times, as claimed.  $\square$

**(3.3) Theorem.** *Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$  and let  $\sigma_k \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  (resp.  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ ) denote any circle traversed  $k \geq 1$  times. Then  $\sigma_k, \sigma_{k+2}$  lie in the same*

<sup>1</sup>By reasoning more carefully it would be possible to avoid using Zorn's lemma.

component of  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  (resp.  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ ) if and only if

$$k \geq \left\lfloor \frac{\pi}{\rho_1 - \rho_2} \right\rfloor \quad (\rho_i = \operatorname{arccot} \kappa_i, \ i = 1, 2).$$

*Proof.* This follows from the combination of (4.4), (8.2) and (8.4), if we use the homeomorphisms in (2.15) and (2.25).  $\square$

**(9.1) Proposition.** *Let  $\kappa_0 = \cot \rho_0 \geq 0$ ,*

$$n = \left\lfloor \frac{\pi}{\rho_0} \right\rfloor + 1.$$

*Then the set  $\mathcal{O}_\nu$  of all condensed curves  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  satisfying  $\nu(\gamma) = \nu$  for some fixed  $\nu \leq n - 2$  is a contractible connected component of  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ .*

*Proof.* When  $\kappa_0 = 0$ , this result is equivalent to the assertion that the component of  $\mathcal{L}_0^{+\infty}(I)$  containing a circle traversed once is contractible; this result is not new, and a proof can be found in [15]. When  $\kappa_0 > 0$ , (6.1) guarantees that  $\mathcal{O}_\nu$  is weakly contractible and, in particular, connected. Proposition (8.4) then implies that  $\mathcal{O}_\nu$  must be a connected component of  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ . Using (2.7(a)) we deduce that  $\mathcal{O}_\nu$  is an open subset of this space. Hence  $\mathcal{O}_\nu$  is also a Hilbert manifold, and it must be contractible by (2.7(b)).  $\square$

*Remark.* Note that if  $\kappa_0 < 0$  (that is, if  $\rho_0 > \frac{\pi}{2}$ ), then it is a consequence of (4.2) and (4.3) that  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$  has only  $n = 2$  components, and the conclusion of (9.1) does not make sense in this case (no curve  $\gamma$  satisfies  $\nu(\gamma) \leq 0$ ). Moreover, these two components are far from being contractible: Even for  $\kappa_0 = -\infty$ , the (co)homology groups of  $\mathcal{J} = \mathcal{L}_{-\infty}^{+\infty}(I) \simeq \Omega \mathbf{S}^3 \sqcup \Omega \mathbf{S}^3$  are non-trivial in infinitely many dimensions.

Our main theorem is a combination of the three previous results.

**(3.1) Theorem.** *Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ ,  $\rho_i = \operatorname{arccot} \kappa_i$  ( $i = 1, 2$ ) and  $[x]$  denote the greatest integer smaller than or equal to  $x$ . Then  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  has exactly  $n$  connected components  $\mathcal{L}_1, \dots, \mathcal{L}_n$ , where*

$$n = \left\lfloor \frac{\pi}{\rho_1 - \rho_2} \right\rfloor + 1$$

*and  $\mathcal{L}_j$  contains circles traversed  $j$  times ( $1 \leq j \leq n$ ). The component  $\mathcal{L}_{n-1}$  also contains circles traversed  $(n - 1) + 2k$  times, and  $\mathcal{L}_n$  contains circles traversed  $n + 2k$  times, for  $k \in \mathbf{N}$ . Moreover, each of  $\mathcal{L}_1, \dots, \mathcal{L}_{n-2}$  is homotopy equivalent to  $\mathbf{SO}_3$  ( $n \geq 3$ ).*

*Proof.* All of the assertions of the theorem but the last one follow from (4.2), (4.3) and the homeomorphism  $\mathcal{L}_{\kappa_1}^{\kappa_2} \approx \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  of (2.15).

Assume that  $n \geq 3$  and let  $\sigma_k \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  be a circle traversed  $k \leq n - 2$  times. In the notation of (9.1), the connected component  $\mathcal{L}_k(I)$  of  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  containing  $\sigma_k$  is mapped to the component  $\mathcal{O}_k$  under the homeomorphism  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I) \approx \mathcal{L}_{\kappa_0}^{+\infty}(I)$  of (2.25), because  $\sigma_k$  is mapped to another circle traversed  $k$  times (cf. (2.23)). Therefore,  $\mathcal{L}_k(I)$  is contractible by (9.1). The last assertion of the theorem is deduced from this and the homeomorphism  $\mathcal{L}_{\kappa_1}^{\kappa_2} \approx \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ .  $\square$

Theorem (4.1) characterizes the connected components of  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  in terms of the circles that they contain. However, a more direct characterization in terms of the properties of a curve is also available.

**(9.2) Theorem.** *Let  $\kappa_0 \in \mathbf{R}$  and let  $\mathcal{L}_1, \dots, \mathcal{L}_n$  be the connected components of  $\mathcal{L}_{\kappa_0}^{+\infty}$ , as described in (4.1). Then  $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$  lies in:*

- (i)  $\mathcal{L}_j$  ( $1 \leq j \leq n - 2$ ) if and only if it is condensed with rotation number  $j$ .
- (ii)  $\mathcal{L}_{n-1}$  if and only if  $\tilde{\Phi}_\gamma(1) = (-1)^{n-1} \tilde{\Phi}_\gamma(0)$  and either it is non-condensed or condensed with rotation number  $\nu(\gamma) \geq n - 1$ .
- (iii)  $\mathcal{L}_n$  if and only if  $\tilde{\Phi}_\gamma(1) = (-1)^n \tilde{\Phi}_\gamma(0)$  and either it is non-condensed or condensed with rotation number  $\nu(\gamma) \geq n - 1$ .

*Proof.* This follows from (4.1) and (9.1).  $\square$

Recall that  $\tilde{\Phi}: [0, 1] \rightarrow \mathbf{S}^3$  is the lift of the frame  $\Phi_\gamma: [0, 1] \rightarrow \mathbf{SO}_3$  of  $\gamma$  to  $\mathbf{S}^3$  (cf. (2.12)). When  $-\infty \leq \kappa_0 < 0$  (resp.  $\rho_1 - \rho_2 > \frac{\pi}{2}$ ) we have  $n = 2$ , and this characterization of the two components  $\mathcal{L}_1, \mathcal{L}_2$  of  $\mathcal{L}_{\kappa_0}^{+\infty}$  (resp.  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ ) may be simplified to:  $\gamma$  lies in  $\mathcal{L}_i$  if and only if  $\tilde{\Phi}_\gamma(1) = (-1)^i \tilde{\Phi}_\gamma(0)$ .

**(9.3) Lemma.** *Let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ ,  $\rho_i = \operatorname{arccot} \kappa_i$  and  $\gamma_i \in \mathcal{L}_{\kappa_1}^{\kappa_2}$  ( $i = 1, 2$ ). Then  $\gamma_1$  lies in the same component of  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  as  $\gamma_2$  if and only if the corresponding translations  $\bar{\gamma}_i$  of  $\gamma_i$  by  $\rho_2$ ,*

$$\bar{\gamma}_i(t) = \cos \rho_2 \gamma_i(t) + \sin \rho_2 \mathbf{n}_i(t) \quad (t \in [0, 1], \ i = 1, 2),$$

*lie in the same connected component of  $\mathcal{L}_{\kappa_0}^{+\infty}$ , where  $\kappa_0 = \cot(\rho_1 - \rho_2)$ .*  $\square$

*Proof.* The proof is immediate, since translation by  $\rho_2$  is a homeomorphism from  $\mathcal{L}_{\kappa_1}^{\kappa_2}$  onto  $\mathcal{L}_{\kappa_0}^{+\infty}$ , as was seen in (2.22).  $\square$

Combining (9.2) and (9.3) we obtain a simple procedure to check whether two curves  $\gamma_1, \gamma_2 \in \mathcal{L}_{\kappa_1}^{\kappa_2}$  lie in the same component of  $\mathcal{L}_{\kappa_1}^{\kappa_2}$ , provided only that we have parametrizations of  $\gamma_1$  and  $\gamma_2$ .



## 10

### The Inclusion $\mathcal{L}_{\kappa_1}^{\kappa_2} \hookrightarrow \mathcal{L}_{-\infty}^{+\infty}$

The objective of this section is to prove that the inclusion  $\mathcal{L}_{\kappa_1}^{\kappa_2} \hookrightarrow \mathcal{L}_{-\infty}^{+\infty}$  is not a homotopy equivalence when

$$\rho_1 - \rho_2 \leq \frac{2\pi}{3},$$

where  $\rho_i = \operatorname{arccot}(\kappa_i)$ ,  $i = 1, 2$ . This section is to a large extent independent of the rest of the work. In particular, we do not use the caustic band, only the regular band. Since we will be working mostly with spaces of type  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}$  (as we are allowed to do, by (2.24)), we start by modifying its definition to suit our needs.

#### The band spanned by a curve

Throughout this subsection, let  $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$  be fixed and let  $\rho_1 = \operatorname{arccot} \kappa_1$ ,  $\rho_2 = \operatorname{arccot} \kappa_2$ . In order to get rid of the distinguished position of the endpoints of  $[0, 1]$ , we shall extend the domain of definition of all (closed) curves  $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  to  $\mathbf{R}$  by declaring them to be 1-periodic.

**(10.1) Definitions.** Let  $\gamma: \mathbf{R} \rightarrow \mathbf{S}^2$ ,  $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ .

(a) The (*regular*) *band*  $B_\gamma$  spanned by  $\gamma$  is the map:

$$B_\gamma: \mathbf{R} \times [\rho_1 - \pi, \rho_2] \rightarrow \mathbf{S}^2, \quad B_\gamma(t, \theta) = \cos \theta \gamma(t) + \sin \theta \mathbf{n}(t). \quad (1)$$

(b)  $B_\gamma$  is *simple* if it is injective when restricted to  $[0, 1] \times [\rho_1 - \pi, \rho_2]$ .

(c)  $B_\gamma$  is *quasi-simple* if it is injective when restricted to  $[0, 1] \times (\rho_1 - \pi, \rho_2)$ .

(d) The *boundary curves* of  $B_\gamma$  are the curves  $\beta_+, \beta_-: \mathbf{R} \rightarrow \mathbf{S}^2$  given by:

$$\beta_+: t \mapsto B_\gamma(t, \rho_2) \quad \text{and} \quad \beta_-: t \mapsto B_\gamma(t, \rho_1 - \pi). \quad (2)$$

Clearly,  $B_\gamma$  is also 1-periodic in  $t$ . Aside from the periodicity, the definition of regular band in (4.6) is subsumed in (10.1). Here are some further basic properties of  $B_\gamma$ .

**(10.2) Lemma.** *Let  $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  and let  $B_\gamma: \mathbf{R} \times [\rho_1 - \pi, \rho_2] \rightarrow \mathbf{S}^2$  be the band spanned by  $\gamma$ . Then:*

(a) *The derivative of  $B_\gamma$  is an isomorphism at every point.*

(b)  *$\frac{\partial B_\gamma}{\partial \theta}(t, \theta)$  has norm 1 and is orthogonal to  $\frac{\partial B_\gamma}{\partial t}(t, \theta)$ . Moreover,*

$$\frac{\partial B_\gamma}{\partial t}(t, \theta) \times \frac{\partial B_\gamma}{\partial \theta}(t, \theta) = \lambda B_\gamma(t, \theta), \text{ with } \lambda > 0.$$

(c) *If  $B_\gamma$  is quasi-simple, then the restriction of  $B_\gamma$  to  $(\rho_1 - \pi, \rho_2)$  is a covering map onto its image.*

*Proof.* The proofs of (a) and (b) are practically identical to those of the corresponding items in (4.7), so they will be omitted. For part (c), consider the unique map  $\bar{B}_\gamma: \mathbf{S}^1 \times (\rho_1 - \pi, \rho_2) \rightarrow \mathbf{S}^2$  making the following diagram commute:

$$\begin{array}{ccc} \mathbf{R} \times (\rho_1 - \pi, \rho_2) & \xrightarrow{B_\gamma} & \mathbf{S}^2 \\ \text{pr} \times \text{id} \downarrow & \nearrow \bar{B}_\gamma & \\ \mathbf{S}^1 \times (\rho_1 - \pi, \rho_2) & & \end{array} \quad (3)$$

where  $\text{pr}(t) = \exp(2\pi it)$ . Since  $\bar{B}_\gamma$  is a diffeomorphism and  $\text{pr} \times \text{id}$  is a covering map,  $B_\gamma$  is also a covering map (onto its image).  $\square$

**(10.3) Lemma.** *Let  $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  and suppose that  $B_\gamma$  is quasi-simple. For fixed  $\varphi$  satisfying  $\rho_1 - \pi < \varphi < \rho_2$ , the curve  $\gamma_\varphi: t \mapsto B_\gamma(t, \varphi)$  separates  $\mathbf{S}^2$  into two connected components, one containing  $B_\gamma(\mathbf{R} \times [\rho_1 - \pi, \varphi])$  and the other containing  $B_\gamma(\mathbf{R} \times (\varphi, \rho_2])$ .*

*Proof.* By (10.2(b)),  $B_\gamma$  is an immersion. Consequently,

$$U = B_\gamma(\mathbf{R} \times (\rho_1 - \pi, \rho_2)) \quad \text{and} \quad U_\varepsilon = B_\gamma(\mathbf{R} \times (\varphi - \varepsilon, \varphi + \varepsilon))$$

are open sets, for any  $\varepsilon > 0$  satisfying  $\varepsilon < \min\{\rho_2 - \varphi, \varphi + \pi - \rho_1\}$ . Let  $S$  denote the image of  $\gamma_\varphi$ . If  $\beta_+(t') \in S$  for some  $t'$ , then  $B_\gamma(t', \theta) \in U_\varepsilon$  for all  $\theta$  close to  $\rho_2$ . This contradicts the fact that  $B_\gamma$  is quasi-simple. Hence  $\gamma_\varphi$  does not intersect  $\beta_+$ , and for the same reason it does not intersect  $\beta_-$  either. Now let

$$A_- = B_\gamma(\mathbf{R} \times [\rho_1 - \pi, \varphi]), \quad A_+ = B_\gamma(\mathbf{R} \times (\varphi, \rho_2]).$$

By the Jordan curve theorem, the simple closed curve  $\gamma_\varphi$  separates  $\mathbf{S}^2$  into two connected components  $V_+$ ,  $V_-$ , and  $S$  is the boundary of each. Let  $V_+$  be the component which contains  $A_+$ . If  $A_- \subset V_+$  then all of  $U \setminus S$  would be contained in  $V_+$ . Since  $U$  is a neighborhood of  $S$ , this would give  $\partial V_- \cap S = \emptyset$ , a contradiction. Hence  $A_- \subset V_-$ .  $\square$

**(10.4) Lemma.** *Let  $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ , with  $B_\gamma$  quasi-simple. If  $B_\gamma(t_1, \theta_1) = B_\gamma(t_2, \theta_2)$  then:*

- (a)  $\theta_1 = \theta_2 \in \{\rho_1 - \pi, \rho_2\}$ .
- (b)  $\frac{\partial B_\gamma}{\partial \theta}(t_2, \theta_2) = -\frac{\partial B_\gamma}{\partial \theta}(t_1, \theta_1)$ ,  $\frac{\partial B_\gamma}{\partial t}(t_2, \theta_2) = -\mu \frac{\partial B_\gamma}{\partial t}(t_1, \theta_1)$ ,  $\mu > 0$ , and  $\mathbf{t}(t_2) = -\mathbf{t}(t_1)$ , unless  $t_1 - t_2 \in \mathbf{Z}$ .

In other words, if  $B_\gamma$  is quasi-simple then all of its self-intersections are either self-intersections of  $\beta_+$  or of  $\beta_-$ , and they are actually points of self-tangency.

*Proof.* Part (a) is an immediate corollary of (10.3). Assume that  $t_1 - t_2 \notin \mathbf{Z}$  and, for the sake of concreteness, that  $\theta_1 = \theta_2 = \rho_2$ . Choose  $\varepsilon > 0$  such that  $(t_1 - \varepsilon, t_1 + \varepsilon) + \mathbf{Z}$  does not intersect  $(t_2 - \varepsilon, t_2 + \varepsilon)$  and let  $U_1$  be the open set

$$U_1 = B_\gamma((t_1 - \varepsilon, t_1 + \varepsilon) \times (\rho_1 - \pi, \rho_2)).$$

If  $\frac{\partial B_\gamma}{\partial t}(t_2, \rho_2)$  is not a positive or negative multiple of  $\frac{\partial B_\gamma}{\partial t}(t_1, \rho_2)$ , then either  $B_\gamma(t_2 + u, \rho_2) \in U_1$  or  $B_\gamma(t_2 - u, \rho_2) \in U_1$  for all sufficiently small  $u > 0$ . This contradicts the fact that  $B_\gamma$  is quasi-simple. Hence

$$\frac{\partial B_\gamma}{\partial t}(t_2, \rho_2) = \pm \mu \frac{\partial B_\gamma}{\partial t}(t_1, \rho_2), \quad \mu > 0, \quad \text{and} \quad \frac{\partial B_\gamma}{\partial \theta}(t_2, \rho_2) = \pm \frac{\partial B_\gamma}{\partial \theta}(t_1, \rho_2),$$

the latter being a consequence of the former, by (10.2(b)). If we had  $\frac{\partial B_\gamma}{\partial \theta}(t_1, \rho_2) = \frac{\partial B_\gamma}{\partial \theta}(t_2, \rho_2)$ , then  $B_\gamma(t_2, \rho_2 - u) \in U_1$  for all sufficiently small  $u > 0$ , again contradicting the fact that  $B_\gamma$  is quasi-simple. Hence

$$\frac{\partial B_\gamma}{\partial \theta}(t_2, \rho_2) = -\frac{\partial B_\gamma}{\partial \theta}(t_1, \rho_2)$$

and (10.2(b)) then yields  $\frac{\partial B_\gamma}{\partial t}(t_2, \rho_2) = -\mu \frac{\partial B_\gamma}{\partial t}(t_1, \rho_2)$ ,  $\mu > 0$ . Together with eq. (6) of §3, this implies that  $\mathbf{t}(t_2) = -\mathbf{t}(t_1)$ .  $\square$

**(10.5) Lemma.** *Let  $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ , with  $B_\gamma$  quasi-simple. Let  $\alpha: [0, 1] \rightarrow \mathbf{S}^2$  be a  $C^1$  curve of length  $L$  such that  $\alpha(0)$  lies in the image of  $\beta_-$  and  $\alpha(1)$  in the image of  $\beta_+$ . Then  $L \geq \pi - (\rho_1 - \rho_2)$ , and equality holds if and only if  $\alpha$  is an orientation-preserving reparametrization of a curve  $\theta \mapsto B_\gamma(t_0, \theta)$ ,  $\theta \in [\rho_1 - \pi, \rho_2]$ .*

More concisely: If a curve crosses the band, it must have length  $\geq \pi - (\rho_1 - \rho_2)$ .

*Proof.* Let

$$t_0 = \sup \{t \in [0, 1] : \alpha(t) \in \beta_-(\mathbf{R})\} \quad \text{and} \quad t_1 = \inf \{t > t_0 : \alpha(t) \in \beta_+(\mathbf{R})\}$$

By (10.4), the images of  $\beta_-$  and  $\beta_+$  do not intersect each other, whence  $t_0 < t_1$ . We lose no generality in assuming that  $t_0 = 0$ ,  $t_1 = 1$ .

Let  $\tau_0 \in [0, 1]$  and  $\theta_0 \in (\rho_1 - \pi, \rho_2)$  be the unique numbers satisfying  $B_\gamma(\tau_0, \theta_0) = \alpha(\frac{1}{2})$ . The image of  $(0, 1)$  by  $\alpha$  is completely contained in  $B_\gamma(\mathbf{R} \times (\rho_1 - \pi, \rho_2))$ , because of the way  $t_0$  and  $t_1$  were chosen. Consequently, by (10.2(c)), there exist unique  $C^1$  functions  $\tau: (0, 1) \rightarrow \mathbf{R}$ ,  $\theta: (0, 1) \rightarrow (\rho_1 - \pi, \rho_2)$  making the following diagram of pointed maps commute:

$$\begin{array}{ccc} ((0, 1), \frac{1}{2}) & \xrightarrow{\alpha} & (\mathbf{S}^2, \alpha(\frac{1}{2})) \\ \tau \times \theta \downarrow & \nearrow B_\gamma & \\ (\mathbf{R} \times (\rho_1 - \pi, \rho_2), (\tau_0, \theta_0)) & & \end{array}$$

The length  $L$  of  $\alpha$  is therefore given by:

$$\begin{aligned} L &= \int_0^1 |\dot{\alpha}(u)| \, du = \lim_{\delta \rightarrow 0^+} \int_\delta^{1-\delta} |\dot{\alpha}(u)| \, du \\ &= \lim_{\delta \rightarrow 0^+} \int_\delta^{1-\delta} \left| \dot{\tau}(u) \frac{\partial B_\gamma}{\partial t}(\tau(u), \theta(u)) + \dot{\theta}(u) \frac{\partial B_\gamma}{\partial \theta}(\tau(u), \theta(u)) \right| \, du \\ &\geq \lim_{\delta \rightarrow 0^+} \int_\delta^{1-\delta} |\dot{\theta}(u)| \, du \geq \lim_{\delta \rightarrow 0^+} \left| \int_\delta^{1-\delta} \dot{\theta}(u) \, du \right| = |\theta(1-) - \theta(0+)|, \end{aligned}$$

where in the first inequality we have used the facts that  $\frac{\partial B_\gamma}{\partial t} \perp \frac{\partial B_\gamma}{\partial \theta}$  and that the latter has norm 1, as proved in (10.2(b)).

We claim that the limits  $\theta(0+)$  and  $\theta(1-)$  exist and are equal to  $\rho_1 - \pi$  and  $\rho_2$ , respectively. Let  $\varphi \in (\rho_1 - \pi, \rho_2)$  be given and let

$$A_- = B_\gamma(\mathbf{R} \times [\rho_1 - \pi, \varphi]) \quad \text{and} \quad A_+ = B_\gamma(\mathbf{R} \times (\varphi, \rho_2]).$$

As we saw in (10.3),  $A_-$  and  $A_+$  are contained in different connected components of  $\mathbf{S}^2 \setminus \gamma_\varphi(\mathbf{R})$ . These components are open sets and  $\alpha(0) \in A_-$  by hypothesis, hence, by continuity, there exists  $\delta > 0$  such that  $\alpha([0, \delta)) \subset A_-$ . This implies that  $\theta(u) < \varphi$  for all  $u \in (0, \delta)$ . Because we can choose  $\varphi$  arbitrarily close to  $\rho_1 - \pi$ , this shows that  $\theta(0+) = \rho_1 - \pi$ . Similarly,  $\theta(1-) = \rho_2$ . Therefore  $L \geq \pi - (\rho_1 - \rho_2)$ .

Furthermore,  $L = \pi - (\rho_1 - \rho_2)$  if and only if  $\dot{\theta}$  does not change sign in  $(0, 1)$  and  $\dot{\tau}(u) = 0$  for all  $u \in (0, 1)$  (recall that, by (10.2(a)),  $\frac{\partial B_\gamma}{\partial t}$  never vanishes). In other words,  $\alpha$  is an orientation-preserving reparametrization of

the curve  $\theta \mapsto B_\gamma(\tau(\frac{1}{2}), \theta)$ ,  $\theta \in [\rho_1 - \pi, \rho_2]$ .  $\square$

### The Topology of $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ for $0 < \kappa_1 \leq \sqrt{3}$

Our next goal is to prove some basic facts about the topology of  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$  for any (fixed)  $\kappa_1$  satisfying  $0 < \kappa_1 \leq \sqrt{3}$ . We shall extend the domain of curves in this space to  $\mathbf{R}$  by declaring them to be 1-periodic.

**(10.6) Definition.** Let  $\mathcal{A}$  denote the subspace of  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$  ( $0 < \kappa_1 \leq \sqrt{3}$ ) consisting of all curves  $\gamma$  such that

$$\gamma(t + \tfrac{1}{2}) = -\gamma(t) \text{ for all } t \in \mathbf{R}. \quad (4)$$

By (10.1), the band of a curve in  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I) \supset \mathcal{A}$  ( $0 < \kappa_1 \leq \sqrt{3}$ ) is defined on

$$\mathbf{R} \times [-\rho_1, \rho_1] \supset \mathbf{R} \times [-\tfrac{\pi}{6}, \tfrac{\pi}{6}].$$

For our purposes it will suffice to consider the restriction of  $B_\gamma$  to the latter set.

**(10.7) Remark.** If  $\gamma \in \mathcal{A}$ , then  $\Phi_\gamma(\frac{1}{2}) = Q_{\mathbf{k}}$ , where

$$Q_{\mathbf{k}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the image of the quaternion  $\mathbf{k}$  (and of  $-\mathbf{k}$ ) under the projection  $\mathbf{S}^3 \rightarrow \mathbf{SO}_3$ . In fact,  $\mathcal{A} \approx \mathcal{L}_{-\kappa_1}^{+\kappa_1}(Q_{\mathbf{k}})$ , because (4) implies that a curve in  $\mathcal{A}$  is uniquely determined by its restriction to  $[0, \frac{1}{2}]$ .

*Remark.* A curve  $\gamma: \mathbf{R} \rightarrow \mathbf{S}^2$  in  $\mathcal{A}$  corresponds to a closed curve  $\bar{\gamma}: \mathbf{S}^1 \rightarrow \mathbf{RP}^2$  induced by  $\gamma$  as follows:

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{\gamma} & \mathbf{S}^2 \\ p \downarrow & & \downarrow q \\ \mathbf{S}^1 & \xrightarrow{\bar{\gamma}} & \mathbf{RP}^2 \end{array}$$

Here  $p(t) = \exp(4\pi it)$  and  $q$  is the usual covering map. Thus, we could also view  $\mathcal{A}$  as a space of closed curves in  $\mathbf{RP}^2$  having curvature bounded by  $\kappa_1$ . (Even though  $\mathbf{RP}^2$  is not orientable, we can still speak of the unsigned curvature of a curve in  $\mathbf{RP}^2$ .) We shall not make any use of this interpretation in the sequel, however.

Examples of curves contained in  $\mathcal{A}$  are the geodesics  $\sigma_m: \mathbf{R} \rightarrow \mathbf{S}^2$  given by:

$$\sigma_m(t) = (\cos((2m+1)2\pi t), \sin((2m+1)2\pi t), 0) \quad (m = 0, 1, 2). \quad (5)$$

One checks directly that

$$\tilde{\Phi}_{\sigma_m}(t) = \cos((2m+1)\pi t) \mathbf{1} + \sin((2m+1)\pi t) \mathbf{k} \quad (m = 0, 1, 2),$$

so that  $\tilde{\Phi}_{\sigma_m}(\frac{1}{2}) = (-1)^m \mathbf{k}$ . Therefore, by (a slightly different version of) lemma (2.13),  $\sigma_1$  does not lie in the same connected component of  $\mathcal{A}$  as  $\sigma_0, \sigma_2$ . We shall see later that  $\sigma_0$  and  $\sigma_2$  do not lie in the same connected component either. In fact, we have the following result.

**(10.8) Proposition.** *Let  $0 < \kappa_1 \leq \sqrt{3}$ . The subspace*

$$\mathcal{A}_0 = \{\gamma \in \mathcal{A} : B_\gamma \text{ is simple}\} \subset \mathcal{A},$$

*which contains  $\sigma_0$  but not  $\sigma_2$ , is both open and closed in  $\mathcal{A}$ .*

To prove this result we will need several lemmas.

**(10.9) Definition.** Let  $\gamma \in \mathcal{A}$ , let  $B_\gamma: \mathbf{R} \times [-\frac{\pi}{6}, \frac{\pi}{6}] \rightarrow \mathbf{S}^2$  be its band and let  $C \subset \mathbf{S}^2$  be a great circle. We shall say that  $[\tau_1, \tau_2] \subset \mathbf{R}$  is a *crossing interval* of  $B_\gamma$  with respect to  $C$  if:

- (i)  $B_\gamma(\{\tau_1\} \times [-\frac{\pi}{6}, \frac{\pi}{6}])$  is contained in a closed disk bounded by  $C$ ;
- (ii)  $B_\gamma(\{\tau_2\} \times [-\frac{\pi}{6}, \frac{\pi}{6}])$  is contained in the other closed disk bounded by  $C$ ;
- (iii)  $B_\gamma(\{t\} \times [-\frac{\pi}{6}, \frac{\pi}{6}])$  is not contained in either of the closed disks bounded by  $C$  for  $t \in (\tau_1, \tau_2)$ .

Thus,  $[\tau_1, \tau_2]$  is a crossing interval if it is a minimal interval during which the band passes from one side of  $C$  to the other. In view of the 1-periodicity of  $B_\gamma$  in  $t$ , we shall identify two crossing intervals which differ by a translation by an integer.

**(10.10) Lemma.** *Let  $\gamma \in \mathcal{A}$ , let  $C \subset \mathbf{S}^2$  be a great circle and  $[\tau_1, \tau_2]$  a crossing interval of  $B_\gamma$ . Then:*

- (a)  $B_\gamma(t + \frac{1}{2}, \theta) = -B_\gamma(t, -\theta)$  for all  $t \in \mathbf{R}$ ,  $\theta \in [-\frac{\pi}{6}, \frac{\pi}{6}]$ . In particular, the images of  $\beta_+$  and  $\beta_-$  are antipodal sets.
- (b)  $[\tau_1 + \frac{1}{2}, \tau_2 + \frac{1}{2}]$  is also a crossing interval.
- (c) Two crossing intervals are either equal or have disjoint interiors.
- (d)  $\tau_2 - \tau_1 \leq \frac{1}{2}$ .

*Proof.* Part (a) follows from definition (10.1) and the relation  $\gamma(t + \frac{1}{2}) = -\gamma(t)$ , which is valid for any  $\gamma \in \mathcal{A}$ .

Since  $C$  is a great circle, the two disks bounded by  $C$  are antipodal sets. Together with (a), this implies that  $[\tau_1 + \frac{1}{2}, \tau_2 + \frac{1}{2}]$  is a crossing interval if  $[\tau_1, \tau_2]$  is, and proves (b).

Part (c) is an immediate consequence of definition (10.9).

Part (d) follows from (b) and (c): If  $\tau_2 - \tau_1 > \frac{1}{2}$ , then  $(\tau_1, \tau_2) \cap (\tau_1 + \frac{1}{2}, \tau_2 + \frac{1}{2}) \neq \emptyset$ .  $\square$

**(10.11) Lemma.** *Let  $\gamma \in \mathcal{A}$ , let  $C \subset \mathbf{S}^2$  be a great circle and let  $[\tau_1, \tau_2]$  be a crossing interval of  $B_\gamma$ . Then the following conditions are equivalent:*

- (i)  $C \cap B_\gamma(\{t\} \times [-\frac{\pi}{6}, \frac{\pi}{6}])$  consists of more than one point for some  $t \in [\tau_1, \tau_2]$ .
- (ii)  $B_\gamma(\{t\} \times [-\frac{\pi}{6}, \frac{\pi}{6}])$  is completely contained in  $C$  for some  $t \in [\tau_1, \tau_2]$ .
- (iii)  $\gamma(t) \in C$  and  $\dot{\gamma}(t)$  is orthogonal to  $C$  for some  $t \in [\tau_1, \tau_2]$ .
- (iv)  $B_\gamma(t, \theta) \in C$  and  $\frac{\partial B_\gamma}{\partial t}(t, \theta)$  is orthogonal to  $C$  for some  $t \in [\tau_1, \tau_2]$  and all  $\theta \in [-\frac{\pi}{6}, \frac{\pi}{6}]$ .
- (v)  $\tau_1 = \tau_2$ .

*Proof.* Suppose that (i) holds, and let  $\Gamma_t$  be parametrized by:

$$u \mapsto \cos u \gamma(t) + \sin u \mathbf{n}(t) \quad (u \in [-\pi, \pi]). \quad (6)$$

By hypothesis, the great circles  $C$  and  $\Gamma_t$  have at least two non-antipodal points in common. Hence, they must coincide, and (ii) holds.

If (ii) holds then  $\frac{\partial B_\gamma}{\partial \theta}(t, 0)$  is tangent to  $C$ . Hence, by (10.2(b)),  $\dot{\gamma}(t) = \frac{\partial B_\gamma}{\partial t}(t, 0)$  is orthogonal to  $C$ , and (iii) holds.

Suppose that (iii) holds. Then  $\frac{\partial B_\gamma}{\partial \theta}(t, 0)$  is tangent to  $C$ , which means that  $C$  and the circle  $\Gamma_t$  defined in (6) are two great circles which are tangent at  $\gamma(t)$ . Therefore  $C = \Gamma_t$ , and  $B_\gamma(\{t\} \times [-\frac{\pi}{6}, \frac{\pi}{6}]) \subset C$ . Since  $\frac{\partial B_\gamma}{\partial t}(t, \theta)$  is a positive multiple of  $\dot{\gamma}(t)$ , it, too, is orthogonal to  $C$ , for every  $\theta \in [-\frac{\pi}{6}, \frac{\pi}{6}]$ .

Suppose now that (iv) holds. Then there exists  $\delta > 0$  such that  $B_\gamma(u, \theta) \notin C$  for  $0 < |u - t| < \delta$  and all  $\theta \in [-\frac{\pi}{6}, \frac{\pi}{6}]$ . This implies that  $\tau_1 = t = \tau_2$ .

Finally, suppose (v) holds and let  $t = \tau_1 = \tau_2$ . Then, according to definition (10.9),  $B_\gamma(\{t\} \times [-\frac{\pi}{6}, \frac{\pi}{6}])$  must be contained in both of the closed disks bounded by  $C$ , that is, it must be contained in  $C$ , whence (i) holds.  $\square$

**(10.12) Lemma.** *Let  $\gamma \in \mathcal{A}$ , with  $B_\gamma$  quasi-simple. Let  $C \subset \mathbf{S}^2$  be a great circle and  $[\tau_1, \tau_2]$  a crossing interval of  $B_\gamma$  (with respect to  $C$ ). Then  $C \cap B_\gamma([\tau_1, \tau_2] \times [-\frac{\pi}{6}, \frac{\pi}{6}])$  has total length  $L \geq \frac{\pi}{3}$ . Moreover, equality holds if and only if  $\tau_1 = \tau_2$ .*

*Proof.* If  $\tau_1 = \tau_2$  then the equivalence (ii) $\leftrightarrow$ (v) in (10.11) shows that  $L = \frac{\pi}{3}$ . Assume now that  $\tau_1 < \tau_2$ . Then, from the equivalence (i) $\leftrightarrow$ (v) in (10.11), we deduce that for each  $t \in [\tau_1, \tau_2]$  there exists exactly one  $\theta(t) \in [-\frac{\pi}{6}, \frac{\pi}{6}]$  such that  $B_\gamma(t, \theta(t)) \in C$ . Again by (10.11),  $\frac{\partial B_\gamma}{\partial t}(t, \theta(t))$  is not orthogonal to  $C$  for any  $t \in [\tau_1, \tau_2]$ . Hence, the implicit function theorem guarantees that  $t \mapsto \theta(t)$  is a  $C^1$  map, and  $\alpha(t) = B_\gamma(t, \theta(t))$  defines a regular curve  $\alpha: [\tau_1, \tau_2] \rightarrow \mathbf{S}^2$ .

Let  $\theta_i = \theta(\tau_i)$ ,  $i = 1, 2$ . We claim first that  $\theta_1, \theta_2 \in \{\pm \frac{\pi}{6}\}$ . Otherwise,  $B_\gamma(\tau_i \times [-\frac{\pi}{6}, \frac{\pi}{6}])$  would contain points on both sides of  $C$ . Further, we claim that  $\theta_2 = -\theta_1$ . Otherwise, say,  $\theta_1 = \theta_2 = -\frac{\pi}{6}$ . If  $\theta(t) \neq \frac{\pi}{6}$  for all  $t \in [\tau_1, \tau_2]$ , then the curve  $t \mapsto B_\gamma(t, \frac{\pi}{6})$  would not cross  $C$  in  $[\tau_1, \tau_2]$ , a contradiction. Let  $\bar{\tau}_2 = \inf \{t \in [\tau_1, \tau_2] : \theta(t) = \frac{\pi}{6}\}$ . Then  $[\tau_1, \bar{\tau}_2] \subset [\tau_1, \tau_2]$  is a crossing interval, hence we must have  $\bar{\tau}_2 = \tau_2$  and  $\theta_2 = \frac{\pi}{6}$ , again a contradiction. Therefore  $\alpha: [\tau_1, \tau_2] \rightarrow \mathbf{S}^2$  is a curve satisfying the hypotheses of (10.5), so it has length  $\geq \frac{\pi}{3}$ , and so does  $C \cap B_\gamma([\tau_1, \tau_2] \times [-\frac{\pi}{6}, \frac{\pi}{6}])$ . The remaining assertion follows from the case of equality in (10.5).  $\square$

**(10.13) Lemma.** *Let  $\gamma_0, \gamma_1 \in \mathcal{A}$  lie in the same connected component, and suppose that  $B_{\gamma_0}$  is simple. Then  $B_{\gamma_1}$  is also simple.*

This result implies that  $\sigma_0$  and  $\sigma_2$  (see eq. (5)) are not in the same connected component. In particular, the number of components of  $\mathcal{A}$  is at least 3. More importantly for us, this lemma implies (10.8):  $\mathcal{A}_0$  is a union of connected components, hence  $\mathcal{A}_0$  is open, and its complement is also a union of connected components, hence  $\mathcal{A}_0$  is closed. (Here we are using the fact that  $\mathcal{A}$  is locally path-connected: As explained in (10.7), it is homeomorphic to the Hilbert manifold  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(Q_{\mathbf{k}})$ .)

*Proof.* Let  $\gamma_s$ ,  $s \in [0, 1]$ , be a path joining  $\gamma_0$  to  $\gamma_1$  in  $\mathcal{A}$ , and let us denote  $B_{\gamma_s}$  simply by  $B_s$ .

We claim first that if  $B_{s_0}$  is simple, then so is  $B_s$  for all  $s$  sufficiently close to  $s_0$ . Indeed, to say that  $B_s$  is simple is the same as to say that the unique map  $\bar{B}_s$  which makes the following diagram commute is injective:

$$\begin{array}{ccc} \mathbf{R} \times [-\frac{\pi}{6}, \frac{\pi}{6}] & \xrightarrow{B_s} & \mathbf{S}^2 \\ p \times \text{id} \downarrow & \nearrow \bar{B}_s & \\ \mathbf{S}^1 \times [-\frac{\pi}{6}, \frac{\pi}{6}] & & \end{array}$$

Here  $p(t) = \exp(2\pi it)$ . Now define

$$f: [0, 1] \times \mathbf{S}^1 \times [-\frac{\pi}{6}, \frac{\pi}{6}] \rightarrow \mathbf{S}^2 \times [0, 1], \quad f(s, z, \theta) = (\bar{B}_s(z, \theta), s).$$



By (10.2(a)),  $\bar{B}_s$  is an immersion for all  $s$ , hence so is  $f$ . Suppose that there exists a sequence  $(s_k)$  with  $s_k \rightarrow s_0$  and  $B_{s_k}$  not simple, and choose  $z_k, z'_k \in \mathbf{S}^1$ ,  $\theta_k, \theta'_k \in [-\frac{\pi}{6}, \frac{\pi}{6}]$  with

$$B_{s_k}(z_k, \theta_k) = B_{s_k}(z'_k, \theta'_k) \quad \text{and} \quad (z_k, \theta_k) \neq (z'_k, \theta'_k) \quad \text{for all } k \in \mathbf{N}.$$

By passing to a subsequence if necessary, we can assume that  $(z_k, \theta_k) \rightarrow (z, \theta)$  and  $(z'_k, \theta'_k) \rightarrow (z', \theta')$ . If  $(z, \theta) \neq (z', \theta')$  then  $\bar{B}_{s_0}$  would not be injective, and if  $(z, \theta) = (z', \theta')$  then  $f$  would not be an immersion. Thus, no such sequence  $(s_k)$  can exist, and this proves our claim.

Now suppose for the sake of obtaining a contradiction that there exists  $s \in [0, 1]$  such that  $B_s$  is not simple, and let  $s_0$  be the infimum of all such  $s$ . From what we have just proved, we know that  $s_0 > 0$  and  $B_{s_0}$  is not simple. We claim that  $B_{s_0}$  is quasi-simple. If not, then there exist  $z_1, z_2 \in \mathbf{S}^1$  and  $\theta_1, \theta_2 \in (-\frac{\pi}{6}, \frac{\pi}{6})$  such that

$$f(s_0, z_1, \theta_1) = f(s_0, z_2, \theta_2) \quad \text{and} \quad (z_1, \theta_1) \neq (z_2, \theta_2).$$

Choose  $\varepsilon > 0$ , open sets  $U_i \ni z_i$  in  $\mathbf{S}^1$  and disjoint neighborhoods  $V_i \ni (s_0, z_i, \theta_i)$  of the form

$$V_i = (s_0 - \varepsilon, s_0] \times U_i \times (\theta_i - \varepsilon, \theta_i + \varepsilon) \quad (i = 1, 2)$$

restricted to which  $f$  is a diffeomorphism. (The fact that  $\theta_i$  belongs to the open interval  $(-\frac{\pi}{6}, \frac{\pi}{6})$  is essential for the definition of  $V_i$ .) Then  $f(s, z, \theta) \in f(V_1)$  for all  $(s, z, \theta) \in V_2$  sufficiently close to  $(s_0, z_2, \theta_2)$ , since  $f(s_0, z_2, \theta_2) \in f(V_1)$ . But this contradicts the fact that  $\bar{B}_s$  is injective for all  $s < s_0$ .

Therefore,  $B_{s_0}$  must be quasi-simple, but not simple. The following lemma shows that this is impossible, which, in turn, allows us to conclude that  $B_s$  must be simple for all  $s \in [0, 1]$ .  $\square$

**(10.14) Lemma.** *Suppose that  $\gamma \in \mathcal{A}$  and  $B_\gamma$  is quasi-simple. Then  $B_\gamma$  is simple.*

*Proof.* If  $p = B_\gamma(t_1, \theta_1) = B_\gamma(t_2, \theta_2)$ ,  $t_1 - t_2 \notin \mathbf{Z}$ , is a point of self-intersection of  $B_\gamma$ , then  $\theta_1 = \theta_2 \in \{\pm\frac{\pi}{6}\}$  and  $\mathbf{t}(t_2) = -\mathbf{t}(t_1)$ , as guaranteed by (10.4).

For  $p$  as above, let  $C_i$  be the circle parametrized by

$$u \mapsto \cos u \gamma(t_i) + \sin u \mathbf{n}(t_i), \quad (u \in [0, 2\pi], \quad i = 1, 2).$$

Then both circles are centered at the origin and pass through  $p$  in a direction orthogonal to  $\mathbf{t}(t_2) = -\mathbf{t}(t_1)$ . Hence  $C_1 = C_2$ , and we shall denote it by  $C$  from

now on. Thus, by (10.10(b)) and (10.11),  $B_\gamma$  has at least the following four crossing intervals, all degenerate:  $\{t_1\}$ ,  $\{t_2\}$ ,  $\{t_1 + \frac{1}{2}\}$  and  $\{t_2 + \frac{1}{2}\}$ . Further, by (10.10(b)), the number of crossing intervals of  $B_\gamma$  is even (or infinite).

Let  $\tau_j \in [0, 1)$ ,  $j = 1, \dots, 4$ , be the numbers  $t_i, t_i + \frac{1}{2} \pmod{1}$  arranged so that  $\tau_j < \tau_{j'}$  if  $j < j'$ . By definition,  $\tau_1, \tau_2 \in [0, \frac{1}{2})$  and  $\tau_3 = \tau_1 + \frac{1}{2}$ ,  $\tau_4 = \tau_2 + \frac{1}{2}$ . Suppose that these are the only crossing intervals of  $B_\gamma$ . Then  $B_\gamma$  crosses from one of the disks  $D_1$  bounded by  $C$  to the other one  $D_2$  at  $t = \tau_1$ , from  $D_2$  to  $D_1$  at  $t = \tau_2$  and from  $D_1$  to  $D_2$  at  $t = \tau_3$ . But the latter is incompatible with  $\dot{\gamma}(\tau_3) = -\dot{\gamma}(\tau_1)$ , which points towards  $D_1$ . We conclude that  $B_\gamma$  has at least six crossing intervals. Since  $C$  has total length  $2\pi$  and  $B_\gamma$  is quasi-simple, (10.11) implies that there cannot be more than six crossing intervals, and that all six are degenerate.

Let us again rearrange the crossing intervals (or numbers)  $\tau_j \in [0, 2)$ ,  $j = 1, \dots, 6$ , so that  $\tau_j < \tau_{j'}$  if  $j < j'$ , and hence  $\tau_i \in [0, \frac{1}{2})$  and  $\tau_{i+3} = \tau_i + \frac{1}{2}$  for  $i = 1, 2, 3$ . The sets  $C_j = B_\gamma(\{\tau_j\} \times [-\frac{\pi}{6}, \frac{\pi}{6}])$  fill out the circle  $C$ , hence  $B_\gamma$  intersects itself in exactly 6 points. Suppose that  $C_1$  and  $C_2$  are disjoint. The image of  $[\tau_1, \tau_2]$  by  $\gamma$  separates the closed disk which contains it in two parts, and the image of  $(\tau_2, \tau_1 + 1)$  by  $\gamma$  contains points in both of these parts. Since  $\gamma$  is a simple curve, this is a contradiction which shows that  $C_j \cap C_{j+1} \neq \emptyset$  for all  $j \pmod{6}$ .

Note that the intersection  $C_j \cap C_{j+1}$  consists of a single point of the form  $B_\gamma(t_j, \theta_{j,j+1}) = B_\gamma(t_{j+1}, \theta_{j,j+1})$ , where  $\theta_{j,j+1} \in \{\pm \frac{\pi}{6}\}$  by (10.4). We may assume without loss of generality that  $\theta_{1,2} = \frac{\pi}{6}$ . This forces  $\theta_{3,4} = \theta_{5,6} = \frac{\pi}{6}$  also.

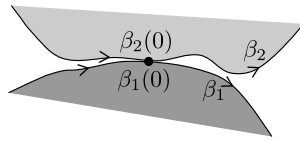


Figure 18: The darkly shaded region consists of points  $B_\gamma(t, \theta)$  for  $(t, \theta)$  close to  $(t_1, \frac{\pi}{6})$ , and the lightly shaded region consists of points  $B_\gamma(t, \theta)$  for  $(t, \theta)$  close to  $(t_2, \frac{\pi}{6})$ . Because  $B_\gamma$  is quasi-simple, the interiors of these regions cannot intersect.

Let  $\rho_j$  denote the radius of curvature of  $\gamma$  at  $\gamma(\tau_j)$ . Then the radius of curvature of  $\beta_+$  at  $t_j$  is  $\rho_j - \frac{\pi}{6}$ , by (2.21). Choose a small  $\varepsilon > 0$  and consider the curves

$$\beta_1, \beta_2: (-\varepsilon, \varepsilon) \rightarrow \mathbf{S}^2, \quad \beta_1(u) = \beta_+(t_1 + u), \quad \beta_2(u) = \beta_+(t_2 - u).$$

Then  $\beta_1(0) = \beta_2(0)$  and  $\dot{\beta}_1(0)$  is a positive multiple of  $\dot{\beta}_2(0)$  by (10.4).

Moreover, the radius of curvature of  $\beta_1$  at 0 is  $\rho_1 - \frac{\pi}{6}$  and that of  $\beta_2$  is  $\pi - (\rho_2 - \frac{\pi}{6}) = \frac{7\pi}{6} - \rho_2$  (the latter formula coming from the reversal of orientation). Because  $B_\gamma$  is quasi-simple,  $\beta_2$  always lies to the right of  $\beta_1$  (with respect to the common tangent unit vector at 0, cf. figure 18), hence the curvature of  $\beta_2$  at 0 is greater than or equal to that of  $\beta_1$  at 0. Or, in terms of the radii of curvature,

$$\rho_1 - \frac{\pi}{6} \geq \frac{7\pi}{6} - \rho_2, \quad \text{that is,} \quad \rho_1 + \rho_2 \geq \frac{4\pi}{3}$$

Similarly,

$$\rho_3 + \rho_4 \geq \frac{4\pi}{3} \quad \text{and} \quad \rho_5 + \rho_6 \geq \frac{4\pi}{3}.$$

Therefore,  $\sum_{j=1}^6 \rho_j \geq 4\pi$ . On the other hand, the relation  $\gamma(t+1) = -\gamma(t)$  yields  $\rho_{i+3} = \pi - \rho_i$ ,  $i = 1, 2, 3$ . Hence  $\sum_{i=1}^6 \rho_i = 3\pi$ . This contradiction shows that the assumption that  $B_\gamma$  has a point of self-intersection, i.e., that  $B_\gamma$  is not simple, must have been false.  $\square$

### The inclusion $\mathcal{L}_{\kappa_1}^{\kappa_2}(I) \hookrightarrow \mathcal{L}_{-\infty}^{+\infty}(I)$

We will show in this subsection that the inclusion

$$i: \mathcal{L}_{\kappa_1}^{\kappa_2}(I) \hookrightarrow \mathcal{L}_{-\infty}^{+\infty}(I) \tag{7}$$

is not a homotopy equivalence when  $0 < \rho_1 - \rho_2 \leq \frac{2\pi}{3}$ , where  $\rho_i = \operatorname{arccot}(\kappa_i)$  (prop. (10.18))

The proof separates into two cases: For  $0 < \rho_1 - \rho_2 \leq \frac{\pi}{2}$ , it is an easy consequence of Little's theorem that  $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$  has at least three connected components, so the map induced by  $i$  on  $\pi_0$  is not a bijection. When  $\frac{\pi}{2} < \rho_1 - \rho_2 \leq \frac{2\pi}{3}$ , both spaces in (7) do have the same number of components, but we will exhibit a non-trivial element of  $\pi_2(\mathcal{L}_{\kappa_1}^{\kappa_2}(I), \gamma_0)$  which lies in the kernel of the induced map  $i_*$  (the basepoint  $\gamma_0$  is a circle traversed once). A very similar construction was previously used by Saldanha in [10] to obtain information on  $\pi_2(\mathcal{L}_0^{+\infty}(I))$  and  $H^2(\mathcal{L}_0^{+\infty}(I))$ .

We conjecture, but do not prove, that the inclusion (7) is not a homotopy equivalence unless  $\rho_1 - \rho_2 = \pi$  (when the inclusion  $i$  is simply the identity map of  $\mathcal{L}_{-\infty}^{+\infty}(I)$ ). In order to show this directly it should be necessary to look at the induced map on  $\pi_{2n}$  for greater and greater  $n$  as  $\rho_1 - \rho_2$  increases to  $\pi$ .

**(10.15) Definition.** Let  $\mathcal{S} \subset \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$  be the image of the map

$$G: (0, 1) \times \mathcal{L}_{-\kappa_1}^{+\kappa_1}(Q_{\mathbf{k}}) \times \mathcal{L}_{-\kappa_1}^{+\kappa_1}(Q_{\mathbf{k}}) \rightarrow \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$$

which associates to a triple  $(t_0, \gamma_1, \gamma_2)$  the curve  $\gamma$  obtained by concatenating  $\gamma_1$  and  $\gamma_2$  at  $t = t_0$ . More precisely,  $G$  is given by:

$$\gamma(t) = G(t_0, \gamma_1, \gamma_2)(t) = \begin{cases} \gamma_1\left(\frac{t}{t_0}\right) & \text{if } 0 \leq t \leq t_0 \\ Q_{\mathbf{k}}\gamma_2\left(\frac{t-t_0}{1-t_0}\right) & \text{if } t_0 \leq t \leq 1 \end{cases}$$

We start by showing that  $\mathcal{S}$  is a submanifold of  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ .

**(10.16) Lemma.** *Let  $\mathcal{S}$  be as above. Then  $\mathcal{S}$  is a closed submanifold of  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$  of codimension 2 which has trivial normal bundle.*

*Proof.* Let  $A$  be the arc of circle

$$A = \left\{ (-\cos \theta, 0, \sin \theta) \in \mathbf{S}^2 : -\frac{\pi}{12} < \theta < \frac{\pi}{12} \right\}.$$

Let  $\mathcal{U}$  be the subset of  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$  consisting of all curves which intersect  $A$  exactly once, and transversally. Then  $\mathcal{U} \supset \mathcal{S}$  and, although  $\mathcal{U}$  is not open, it is a neighborhood of  $\mathcal{S}$ . Given  $\gamma \in \mathcal{U}$ , there exists exactly one  $t_\gamma \in (0, 1)$  such that  $\gamma(t_\gamma) \in A$ . Write

$$\Phi_\gamma(t_\gamma) = \begin{pmatrix} -\cos \theta_\gamma & * & * \\ 0 & * & * \\ \sin \theta_\gamma & z_\gamma & * \end{pmatrix},$$

so that  $\theta_\gamma$  marks the point where  $\gamma$  crosses  $A$  and  $z_\gamma$  measures the slope of the crossing at this point. Define a map  $F: \mathcal{U} \rightarrow \mathbf{R}^2$  by

$$F(\gamma) = (\theta_\gamma, z_\gamma).$$

Then  $\mathcal{S} = F^{-1}(0, 0)$ , and it is easy to see that  $F$  is a submersion at any point of  $\mathcal{S}$ . Hence, lemma (2.7(c)) applies.  $\square$

**(10.17) Lemma.** *Let  $1 < \kappa_1 \leq \sqrt{3}$ . Then there exists  $f: \mathbf{S}^2 \rightarrow \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$  such that:*

(i)  *$f$  intersects  $\mathcal{S}$  only once and transversally;*

(ii)  *$f$  is null-homotopic in  $\mathcal{L}_{-\infty}^{+\infty}(I) \supset \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ .*

*Proof.* Let  $\sigma_\alpha$ ,  $0 \leq \alpha \leq \pi$ , be as described on pp. 83–84 and illustrated in figure 16. Since

$$\kappa_1 > 1 = \tan\left(\frac{\pi}{4}\right),$$

we may define a map  $g: \mathbf{S}^2 \rightarrow \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$  as follows: Set

$$g(N)(t) = (\cos 2\pi t, \sin 2\pi t, 0), \quad g(-N)(t) = (\cos 6\pi t, \sin 6\pi t, 0) \quad (t \in [0, 1])$$

and, for  $p \neq \pm N$ , write  $p = (\cos \theta \sin \alpha, \sin \theta \sin \alpha, \cos \alpha)$  with  $\theta \in [0, 2\pi]$ ,  $\alpha \in (0, \pi)$ . Set

$$g(p)(t) = \left( \Phi_{\sigma_\alpha} \left( t - \frac{\theta}{4\pi} \right) \right)^{-1} \sigma_\alpha \left( t - \frac{\theta}{4\pi} \right) \quad (t \in [0, 1], p \neq \pm N).$$

Thus, any longitude circle  $\theta = \theta_0$  describes a homotopy between a circle traversed once and a circle traversed three times in  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ ; as  $\theta_0$  varies, the only thing that changes is the starting point of the curves in homotopy, and we use multiplication by  $\Phi_{\sigma_\alpha}^{-1}$  to ensure that all curves have the correct frames.

To define  $f: \mathbf{S}^2 \rightarrow \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$  as in the statement, let  $r: \mathbf{S}^2 \rightarrow \mathbf{S}^2$  be reflection across the  $yz$ -plane, and let  $\gamma_2 \in \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$  be the equator traversed two times. Define

$$\bar{g}: \mathbf{S}^2 \rightarrow \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I), \quad \bar{g}(p) = \gamma_2 * ((g \circ r)(p)),$$

where  $*$  denotes the concatenation of paths. Then  $[\bar{g}] = -[g]$  in  $\pi_2(\mathcal{L}_{-\infty}^{+\infty}(I), \sigma_0)$ , because  $[g \circ r] = -[g]$  and concatenating with  $\gamma_2$  has no effect on the homotopy class: For any map  $h: K \rightarrow \mathcal{L}_{-\infty}^{+\infty}(I)$  with domain a compact set,  $h$  and  $\gamma_2 * h$  are homotopic.

Therefore, if we define  $f: \mathbf{S}^2 \rightarrow \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$  to be the concatenation of  $g$  and  $\bar{g}$  (as in the sum operation in  $\pi_2$ ), then trivially  $[f] = 0$ . Moreover, it is immediate from the definition of  $\mathcal{S}$  that  $f(p) \in \mathcal{S}$  if and only if  $p = N$ .  $\square$

**(10.18) Proposition.** *The inclusion*

$$i: \mathcal{L}_{\kappa_1}^{\kappa_2}(I) \hookrightarrow \mathcal{L}_{-\infty}^{+\infty}(I) \simeq \Omega \mathbf{S}^3 \sqcup \Omega \mathbf{S}^3 \quad (8)$$

*is not a weak homotopy equivalence for  $0 < \rho_1 - \rho_2 \leq \frac{2\pi}{3}$ , where  $\rho_i = \operatorname{arccot} \kappa_i$ .*

*Proof.* If  $\kappa_0 \geq 0$ , then  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$  is a subspace of  $\mathcal{L}_0^{+\infty}(I)$ . Let  $\sigma_j \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$  be a circle traversed  $j$  times ( $j = 1, 2, 3$ ). Little's theorem guarantees that  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  are in pairwise distinct components of  $\mathcal{L}_0^{+\infty}(I)$ . Consequently, they must also be in different components of  $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ . Together with (2.22), this implies that the map induced by (8) on  $\pi_0$  is not a bijection for  $0 < \rho_1 - \rho_2 \leq \frac{\pi}{2}$ .

For the remaining cases we work instead with spaces of type  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$  ( $1 < \kappa_1 \leq \sqrt{3}$ ). It suffices to show that the map induced by (8) is not an isomorphism on  $\pi_2$  in this case. Let  $\mathcal{L} = \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ , and let  $\mathcal{S}$  be its submanifold described in (10.15)

By (10.16), the normal bundle  $N\mathcal{S}$  of  $\mathcal{S}$  in  $\mathcal{L}$  is trivial, hence orientable. Let  $\tau$  be a 2-form representing the Thom class of this bundle. Using a tubular neighborhood  $\mathcal{T}$  of  $\mathcal{S}$  in  $\mathcal{L}$ , we can assume that  $\tau$  is a 2-form defined on  $\mathcal{T}$ , extended by 0 to all of  $\mathcal{L}$ . Let  $f: \mathbf{S}^2 \rightarrow \mathcal{L}$  be the map constructed in (10.17).

Then  $f^*\tau$  is a 2-form on  $\mathbf{S}^2$  which represents the Thom class of the normal bundle of  $f^{-1}(\mathcal{S})$  in  $\mathbf{S}^2$ .

Now let  $S$  be a an oriented submanifold of an oriented, finite-dimensional manifold  $M$ . Then the Poincaré dual of  $S$  and the Thom class of the normal bundle of  $S$  in  $M$  are represented by the same form (see [1], pp. 66–67). Applying this to  $M = \mathbf{S}^2$  and  $S = f^{-1}(\mathcal{S})$ , we obtain that  $f^*\tau$  represents the Poincaré dual in  $\mathbf{S}^2$  of a point. Therefore:

$$\int_{\mathbf{S}^2} f^*\tau = 1.$$

In particular, we conclude that  $f$  cannot be null-homotopic in  $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ , otherwise  $f^* = 0$ . As we saw in (10.17),  $f$  is null-homotopic in  $\mathcal{L}_{-\infty}^{+\infty}(I)$ , whence

$$i_*: \pi_2(\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I), \gamma_0) \rightarrow \pi_2(\mathcal{L}_{-\infty}^{+\infty}(I), \gamma_0) \quad (1 < \kappa_1 \leq \sqrt{3})$$

is not injective, where  $\gamma_0$  a circle traversed once, as in (10.17). □

## 11

### Basic Results on Convexity

In this section we collect some results on convexity, none of which is new, that are used throughout the work.

Let  $C \subset \mathbf{R}^{n+1}$ . We say that  $C$  is *convex* if it contains the line segment  $[p, q]$  joining  $p$  to  $q$  whenever  $p, q \in C$ . The *convex hull*  $\hat{X}$  of a subset  $X \subset \mathbf{R}^{n+1}$  is the intersection of all convex subsets of  $\mathbf{R}^{n+1}$  which contain  $X$ . It may be characterized as the set of all points  $q$  of the form

$$q = \sum_{k=1}^m s_k p_k, \quad \text{where} \quad \sum_{k=1}^m s_k = 1, \quad s_k > 0 \quad \text{and} \quad p_k \in X \quad \text{for each } k. \quad (1)$$

**(11.1) Lemma.** *Let  $X \subset \mathbf{S}^n$  and consider the conditions:*

- (i) *0 does not belong to the closure of  $\hat{X}$ .*
- (ii) *There exists an open hemisphere containing  $X$ .*
- (iii) *0 does not belong to  $\hat{X}$ .*
- (iv)  *$X$  does not contain any pair of antipodal points.*

*Then (i)  $\rightarrow$  (ii)  $\rightarrow$  (iii)  $\rightarrow$  (iv), but none of the implications is reversible. If  $X$  is closed then (ii) and (iii) are equivalent.*

*Proof.*

- (i)  $\rightarrow$  (ii) This is a special case of the Hahn-Banach theorem, since  $\{0\}$  is a compact convex set and the closure of  $\hat{X}$  is a closed convex set.
- (ii)  $\nrightarrow$  (i) For  $X \subset \mathbf{S}^n$  the open upper hemisphere, we have

$$\hat{X} = \{(x_1, \dots, x_{n+1}) \in \mathbf{D}^{n+1} : x_{n+1} > 0\}.$$

Hence the closure of  $\hat{X}$  contains the origin, even though  $X$  is (contained in) an open hemisphere.

- (ii)  $\rightarrow$  (iii) Let  $H = \{p \in \mathbf{S}^n : \langle p, h \rangle > 0\}$  be an open hemisphere containing  $X$  and  $U = \{p \in \mathbf{R}^{n+1} : \langle p, h \rangle > 0\}$ . Then  $U$  is convex,  $X \subset U$  and  $0 \notin U$ . Thus,  $0 \notin \hat{X}$ .

- (iii)  $\not\rightarrow$  (ii) Let  $X$  be the image of  $[0, \pi)$  under  $t \mapsto \exp(it)$ .
- (iii)  $\rightarrow$  (iv) If  $p$  and  $-p$  both belong to  $X$ , then  $0 \in [-p, p] \subset \hat{X}$ .
- (iv)  $\not\rightarrow$  (iii) Let  $X = \{1, \zeta, \zeta^2\} \subset \mathbf{S}^1$ , where  $\zeta = \exp(\frac{2}{3}\pi i)$  is a primitive third root of unity. Then  $X$  does not contain antipodal points, but  $0 = \frac{1}{3}(1 + \zeta + \zeta^2)$ .  $\square$

The last assertion is the combination of (i)  $\rightarrow$  (ii) and (ii)  $\rightarrow$  (iii), together with the fact that  $\hat{X}$  is closed if  $X$  is closed, as shown in (11.6) below (its proof does not rely on the present lemma).

**(11.2) Lemma.** *Let  $X \subset \mathbf{S}^n$ . Then 0 belongs to the interior of  $\hat{X}$  if and only if  $X$  is not contained in any closed hemisphere of  $\mathbf{S}^n$ .*

*Proof.* Suppose first that  $0 \notin \text{Int } \hat{X}$ . If 0 does not belong to the closure of  $\hat{X}$  then, as above, we can use the Hahn-Banach theorem to find a hyperplane separating 0 and  $X$ . If  $0 \in \partial \hat{X}$  then there exists a supporting hyperplane for  $\hat{X}$  through 0. One of the closed hemispheres determined by this hyperplane contains  $X$ .

Conversely, if  $X$  is contained in a closed hemisphere

$$H = \{p \in \mathbf{S}^n : \langle p, h \rangle \geq 0\}$$

then  $\hat{X}$  is contained in the “dome”

$$D = \{p \in \mathbf{R}^n : |p| \leq 1 \text{ and } \langle p, h \rangle \geq 0\},$$

which contains 0 in its boundary. Hence,  $\hat{X}$  cannot contain 0 in its interior.  $\square$

Let  $A \subset \mathbf{S}^n$ ,  $n \geq 1$ . We say that  $A$  is *geodesically convex* if it contains no antipodal points and if for any  $p, q \in A$ , the shortest geodesic joining  $p$  to  $q$  is also contained in  $A$ . The *convexification*  $\check{X}$  of a subset  $X \subset \mathbf{S}^n$  is defined to be the intersection of all geodesically convex subsets of  $\mathbf{S}^n$  which contain  $X$ ; if no such subset exists, then we set  $\check{X} = \mathbf{S}^n$ .

In what follows let  $\text{pr}: \mathbf{R}^{n+1} \setminus \{0\} \rightarrow \mathbf{S}^n$  denote the gnomonic projection  $x \mapsto \frac{x}{|x|}$ .

**(11.3) Lemma.** *Let  $X \subset \mathbf{S}^n$ .*

(a) *If  $0 \notin \hat{X}$  then  $\check{X} = \text{pr}(\hat{X})$ .*

(b)  *$0 \in \hat{X}$  if and only if  $\check{X} = \mathbf{S}^n$ .*

*Proof.* We may assume that  $X \neq \emptyset$  since (a) and (b) are trivially true if  $X = \emptyset$ .



- (a) Assume that  $0 \notin \hat{X}$ . If  $p = \text{pr}(p_0)$  and  $-p = \text{pr}(p'_0)$  for  $p_0, p'_0 \in \hat{X}$ , then  $0 \in [p_0, p'_0] \subset \hat{X}$ , a contradiction. Hence,  $\text{pr}(\hat{X})$  does not contain any antipodal points.

Let  $q \in \hat{X}$  be as in (1). We shall prove by induction on  $m$  that  $\text{pr}(q) \in \check{X}$ . This is obvious for  $m = 1$ , so assume  $m > 1$ , and set  $\sigma = s_1 + \cdots + s_{m-1}$ . Then

$$q = (1 - \sigma)p_1 + \sigma \left( \sum_{k=2}^m \frac{s_k}{\sigma} p_k \right) = (1 - \sigma)p_1 + \sigma p.$$

Both  $p_1$  and  $p$  belong to  $\hat{X}$ . Moreover, by the induction hypothesis,  $\text{pr}(p) \in \check{X}$ . Since  $\check{X}$  is geodesically convex, it contains the shortest geodesic joining  $p_1$  to  $\text{pr}(p)$ , which is precisely the image of the line segment  $[p_1, p]$  under  $\text{pr}$ . Hence  $\text{pr}(q) \in \check{X}$ , and  $\text{pr}(\hat{X}) \subset \check{X}$  is established.

Now let  $p = \text{pr}(p_0)$ ,  $q = \text{pr}(q_0)$ , with  $p_0, q_0 \in \hat{X}$ . Then  $(1 - s)p_0 + sq_0 \in \hat{X}$  for all  $s \in [0, 1]$ , whence  $\text{pr}[p_0, q_0] \subset \text{pr}(\hat{X})$ . Since  $\text{pr}[p_0, q_0]$  is the shortest geodesic joining  $p$  to  $q$ , we conclude that  $\text{pr}(\hat{X})$  is geodesically convex. Therefore the reverse inclusion  $\check{X} \subset \text{pr}(\hat{X})$  also holds.

- (b) Suppose first that  $0 \in \hat{X}$  and write  $0$  as a convex combination

$$0 = \sum_{k=1}^m s_k p_k, \quad \text{where} \quad \sum_{k=1}^m s_k = 1, \quad p_k \in X \quad \text{and} \quad s_k > 0 \quad \text{for each } k,$$

with  $m$  as small as possible; clearly,  $m > 1$ . Set  $\sigma = s_2 + \cdots + s_m$ . Then

$$0 = (1 - \sigma)p_1 + \sigma \left( \sum_{k=2}^m \frac{s_k}{\sigma} p_k \right) = (1 - \sigma)p_1 + \sigma p.$$

Let  $S = \{p_2, \dots, p_m\} \subset \mathbf{S}^n$ . If  $0 \in \hat{S}$ , then we would be able to write  $0$  as a convex combination of  $m - 1$  points in  $X$ , a contradiction. Thus, applying part (a) to  $S$ , we deduce that  $\text{pr}(p) \in \check{S} \subset \check{X}$ . Because  $0 \in [p_1, p]$ ,  $p_1$  and  $\text{pr}(p)$  must be antipodal to each other, whence  $\check{X}$  contains a pair of antipodal points. Therefore  $\check{X} = \mathbf{S}^n$ .

Finally, suppose that  $0 \notin \hat{X}$ . By part (a),  $\check{X} = \text{pr}(\hat{X})$ . Further, as we saw in the first paragraph of the proof,  $\text{pr}(\hat{X})$  does not contain antipodal points. Therefore  $\check{X} \neq \mathbf{S}^n$ .  $\square$

**(11.4) Lemma.** *A convex set  $C \subset \mathbf{R}^n$  has empty interior if and only if it is contained in a hyperplane.*

*Proof.* Suppose that  $C$  is not contained in a hyperplane and let  $x_0 \in C$ . Then we can find  $x_1, \dots, x_n \in C$  such that  $\{x_i - x_0\}_{i=1, \dots, n}$  forms a basis for  $\mathbf{R}^n$ .

Being convex,  $C$  must contain the simplex  $[x_0, \dots, x_n]$ , which has nonempty interior since it is homeomorphic to the standard  $n$ -simplex. The converse is obvious.  $\square$

**(11.5) Lemma.** *Let  $X \subset \mathbf{R}^n$  be any set. If  $p \in \hat{X}$ , then there exists a  $k$ -dimensional simplex which has vertices in  $X$  and contains  $p$ , for some  $k \leq n$ .*

Another way to formulate this result is the following: If  $X \subset \mathbf{R}^n$  and  $p \in \hat{X}$ , then it is possible to write  $p$  as a convex combination of  $k + 1$  points in  $X$  which are in general position, where  $k$  is at most equal to  $n$ .

*Proof.* If  $p \in \hat{X}$  then  $p$  can be written as a finite convex combination of points in  $X$ . Hence, we may always assume that  $X$  is finite. The proof will be by induction on  $m + n$ , where  $m$  is the cardinality of  $X$ . If  $m = 1$  or  $n = 1$  the result is trivial.

Let  $X = \{x_0, \dots, x_m\}$  and  $X_0 = X \setminus \{x_0\}$ . If  $p \in \hat{X}_0$  then we can use the induction hypothesis on  $X_0$ , so we may suppose that  $p \notin \hat{X}_0$ . There exist  $q \in \hat{X}_0$  and  $t \in [0, 1]$  such that  $p = (1 - t)x_0 + tq$ . Let  $t_0$  be the infimum of all  $u \geq t$  such that  $(1 - u)x_0 + uq \in \hat{X}_0$ , and let  $q_0 = (1 - t_0)x_0 + t_0q$  be the corresponding point. Note that  $q_0 \in \hat{X}_0$  since the latter set is closed, by (11.6), and that  $t_0 > t$ , since  $p \notin \hat{X}_0$ .

If  $X_0$  is contained in some hyperplane, then we can apply the induction hypothesis to  $X_0 \subset \mathbf{R}^{n-1}$  to conclude that there exists some  $(k-1)$ -dimensional simplex  $\Delta_0$  with vertices in  $X_0$  containing  $q_0$ , for some  $k \leq n$ . Then  $p$  belongs to the  $k$ -dimensional simplex which is the cone on  $\Delta_0$  with vertex  $x_0$ .

If  $X_0$  is not contained in a hyperplane then  $\hat{X}_0$  has nonempty interior in  $\mathbf{R}^n$ , by (11.4). Suppose that it is not possible to write  $q_0$  as a combination of fewer than  $m$  points in  $X_0$ . Then  $q_0 \in \text{Int } \hat{X}_0$ , so that  $(1 - t)x_0 + tq \in \hat{X}_0$  for all  $t$  sufficiently close to  $t_0$ . This contradicts the choice of  $t_0$ . Hence, we may write  $q_0$  as a convex combination of  $m - 1$  points in  $X_0$ , and  $p$  as a convex combination of  $m$  points in  $X$ . By the induction hypothesis, we conclude that  $p$  lies in some  $k$ -dimensional simplex with vertices in  $X$ ,  $k \leq n$ .  $\square$

Let  $Y \subset \mathbf{R}^2$  the graph of the function  $f(t) = (1 + t^2)^{-1}$ , for  $t \in \mathbf{R}$ . Then any point on the  $x$ -axis belongs to the closure of  $\hat{Y}$ , but not to  $\hat{Y}$ . Thus, even though  $Y$  is closed,  $\hat{Y}$  is not. When  $X$  is compact, however, the situation is different.

**(11.6) Lemma.** *If  $X \subset \mathbf{R}^n$  is compact, then  $\hat{X}$  is compact. In particular, if  $X \subset \mathbf{S}^n$  is closed, then  $\hat{X}$  is compact.*  $\square$

*Proof.* Let

$$\Delta^n = \{(s_1, \dots, s_{n+1}) \in \mathbf{R}^{n+1} : s_1 + s_2 + \dots + s_{n+1} = 1, \ s_i \in [0, 1] \text{ for all } i\}$$

and define  $f: \Delta^n \times X^{n+1} \rightarrow \mathbf{R}^n$  by

$$f(s_1, \dots, s_{n+1}, x_1, \dots, x_{n+1}) = s_1x_1 + s_2x_2 + \dots + s_{n+1}x_{n+1}.$$

By (11.5), the image of  $f$  is exactly the convex closure  $\hat{X}$  of  $X$ . Since  $\Delta^n \times X^{n+1}$  is compact and  $f$  is continuous,  $\hat{X}$  must also be compact.  $\square$

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